EIGENCURVES IN THE WORLD OF MODULAR FORMS

Agenda :

1) Define the p-adic Hecke algebra

2) Construct "weight space"

3) Give an e.g. of a family of systems of Hecke eigenvalues parametrized by weight.

4) Define the Hida family & the eigencurve (Coleman & Mazur)

5) Give idea of proofs of Hida's & C&M's theorem.

§1. p-adic Hecke algebra

Fix $N \geq 1$ Fix $p \nmid N$

$M_k(N) \longrightarrow$ wt k, level N for $\Gamma_1(N)$

If $\ell \nmid N$, define S_ℓ to be $\langle \ell \rangle \ell^{k-2}$
for ℓ prime

- $\mathbb{H}_k$ = $\mathbb{Z}$- subalgebra of $\mathrm{End}(M_k(N))$ generated by ℓS_ℓ & T_ℓ as ℓ ranges over primes not

$\hookrightarrow \Gamma_1(N) \begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix} \Gamma_1(N)$

$$\Gamma_0(N)/\Gamma_1(N) \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^\times$$

$$\alpha \longrightarrow \ell$$

$$\langle \ell \rangle = \Gamma_1(N)\, \alpha\, \Gamma_1(N)$$

- $\lambda : \mathbb{H}_k \xrightarrow{\text{rg hom}} \mathbb{C}$ is called a "system of Hecke eigenvalues".

  $\mathbb{H}_k \to \overline{\mathbb{Z}} \to \mathbb{C}$

  We think of λ as taking values in $\overline{\mathbb{Z}}_p$

- $\mathbb{H}_k^{(p)}$ is the subalgebra of $\mathbb{H}_k$ generated by ℓS_ℓ & T_ℓ for ℓ not dividing Np. (p ← prime, fixed; N ← fixed, level)

Define S_ℓ & T_ℓ on $\bigoplus_{i=1}^{m} M_i(N)$ in the obvious way.
S_ℓ acts on $M_i(N)$ via $\langle \ell \rangle \ell^{i-2}$.

- Let $\mathbb{H}^{(p)}_{\leq k} := \mathbb{Z}$-subalgebra of endomorphisms of $\bigoplus_{i=1}^{k} M_i(N)$ generated by ℓS_ℓ & T_ℓ for $\ell \nmid Np$.

  $\mathbb{H}^{(p)}_{\leq k} \hookrightarrow \prod_{i=1}^{k} \mathbb{H}_i \subset \prod \operatorname{End}(M_i(N))$

  $\ell S_\ell \mapsto (\ell S_\ell)_i$

  If $k' \geq k$, then $\bigoplus_{i=p}^{k} \cdots \subset \bigoplus_{i=0}^{k'} \cdots$

  $$\Rightarrow \mathbb{H}^{(p)}_{\leq k'} \xrightarrow{\text{restriction}} \mathbb{H}^{(p)}_{\leq k}$$
  $$\ell S_\ell \longmapsto \ell S_\ell$$
  $$T_\ell \longmapsto T_\ell$$

  $$\Rightarrow \mathbb{Z}_p \otimes \mathbb{H}^{(p)}_{\leq k'} \twoheadrightarrow \mathbb{Z}_p \otimes \mathbb{H}^{(p)}_{\leq k}$$

  $S_\ell \in$ ($\mathbb{Z}_p$ inverts ℓ)

- The <u>p-adic Hecke algebra</u> $\mathbb{H}$ is $:= \varprojlim \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{H}^{(p)}_{\leq k} \hookrightarrow \prod \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{H}_k$

  $\exists$ a well defined S_ℓ & $T_\ell \in \mathbb{H}$

Note:

$$\mathbb{H} \hookrightarrow \prod_{k \geq 1} \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{H}_k \twoheadrightarrow \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{H}_k$$

(Fact : $\mathbb{H}$ is t.f., p-adically complete, Noeth $\mathbb{Z}_p$-algebra & it is a product of finitely many complete Noeth local $\mathbb{Z}_p$ algebras)

- A <u>p-adic system of Hecke eigenvalues</u> is a $\mathbb{Z}_p$-hom $\xi : \mathbb{H} \longrightarrow \bar{\mathbb{Z}}_p$

  consider $\lambda^{(P)} : \mathbb{H}_k^{(P)} \hookrightarrow \mathbb{H}_k \xrightarrow{\lambda} \bar{\mathbb{Z}}_p$

  Then ξ of the form $\lambda^{(P)} \circ (\mathbb{H} \twoheadrightarrow \mathbb{Z}_p \otimes \mathbb{H}_k^{(P)})$ is called <u>classical</u>

§2. Weight Space

$\exists$ a canonical map $\operatorname{Spec} \mathbb{H} \longrightarrow \underbrace{\operatorname{Spec} \mathbb{Z}_p[[T]]}_{\text{This will be our weight space}}$

Let $a := \begin{cases} p & \text{if } p \text{ is odd} \\ p^2 & \text{if } p \text{ is even} \end{cases}$

$\mathbb{Z}_p^\times \cong \mu_{p-1} \times \Gamma$ for p odd

$\mathbb{Z}_p^\times \cong \mu_2 \times \Gamma$ for p even

let $\Gamma := 1 + a\mathbb{Z}_p$ ($U^{(1)}$ if p is odd)

let $\mathbb{Z}_p[[\Gamma]] := \varprojlim_n \mathbb{Z}_p[\underbrace{\Gamma/\Gamma^{p^n}}_{\downarrow \log \; p\mathbb{Z}_p/p^{n+1}\mathbb{Z}_p}]$, the completed gp rg of Γ over $\mathbb{Z}_p$

We have $\mathbb{Z}_p[\Gamma] \hookrightarrow \mathbb{Z}_p[[\Gamma]]$

Denote with $[x]$ the elt in $\mathbb{Z}_p[[\Gamma]]$ corresponding to $x \in \Gamma$

$$\mathbb{Z}_p[[T]] \xrightarrow{\sim} \mathbb{Z}_p[[\Gamma]]$$
$$T \longmapsto -1[1] + 1[1+q]$$

Want to construct $\mathbb{Z}_p[[\Gamma]] \longrightarrow \mathbb{H}$

Suffices to construct a continuous gp hom $\Gamma \longrightarrow \mathbb{H}^\times$

$$\ell \mapsto S_\ell \to \ell^{k-2}\langle \ell \rangle$$

Consider the following set (dense in Γ) : by Dirichlet's theorem on primes in arithmetic progression

$$\mathcal{L} := \{\ell \text{ prime} \mid \ell \equiv 1 \bmod Nq\}$$

p when p is odd

Lemma : The map $\mathcal{L} \longrightarrow \mathbb{H}$ given by $\ell \mapsto S_\ell$, extends uniquely to a continuous gp hom $\Gamma \longrightarrow \mathbb{H}^\times$

Pf :

$$\mathcal{L} \longrightarrow \varprojlim \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{H}^{(p)}_{\leq k} \hookrightarrow \prod_k \mathbb{Z}_p \otimes \mathbb{H}_k^{(p)}$$

$$\mathbb{H} = \varprojlim \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{H}^{(p)}_{\leq k}$$

$$\ell \longmapsto S_\ell \longmapsto (S_\ell)_k = (\ell^{k-2})_k$$

$\mathbb{H}_k = \mathbb{Z}\langle \ell S_\ell, T_\ell \rangle_{\ell \text{ prime not dividing } N}$

$\ell \equiv 1 \bmod Nq$, $\langle \ell \rangle$ is trivial

$p \nmid N$

$\Gamma_1(N)$

This extends to a cont hom on Γ

$$x \longmapsto (x^{k-2})_k$$

As $\mathbb{H}$ is a complete subspace & $\mathcal{L}$ is dense, the image of Γ lands in $\mathbb{H}$

We obtain :

$$-1 + [1+\alpha] \;\longleftarrow\; T$$

$$\operatorname{Spec} \mathbb{H} \xrightarrow{w} \operatorname{Spec} \mathbb{Z}_p[[\Gamma]] \xrightarrow{\sim} \operatorname{Spec} \mathbb{Z}_p[[T]]$$

$\overline{\mathbb{Z}}_p$ points of $\mathbb{Z}_p[[\Gamma]] \longleftrightarrow$ cont. characters $\kappa : \Gamma \to \overline{\mathbb{Z}}_p^{\times}$

$$\operatorname{Spec} \mathbb{Z}_p[[\Gamma]]\,(\overline{\mathbb{Z}}_p) \xrightarrow{\sim} \operatorname{Spec} \mathbb{Z}_p[[T]]\,(\overline{\mathbb{Z}}_p)$$

$$\kappa \longmapsto \kappa(1+\alpha) - 1$$

$$\operatorname{Spec} \overline{\mathbb{Z}}_p \longrightarrow \operatorname{Spec} \mathbb{Z}_p[[T]]$$

$\mathbb{Z}_p[[T]]$

$\mathbb{Z}_p[[\Gamma]] \hookleftarrow 1+p\mathbb{Z}_p$

$\varprojlim_n \mathbb{Z}_p[\Gamma/\Gamma^{p^n}]$

Define $\kappa_k : \Gamma \longrightarrow \overline{\mathbb{Z}}_p^{\times}$ via

$$x \longmapsto x^{k-2}$$

$\{\kappa_k\}$ are Zariski dense in $\operatorname{Spec} \mathbb{Z}_p[[\Gamma]]$

We call κ_k the pt of wt k

We will regard $\operatorname{Spec} \mathbb{Z}_p[[\Gamma]]$ as an interpolation of the set of integers

$$\varprojlim_n \mathbb{H}^{(p)}_{\le k} \otimes \mathbb{Z}_p$$

Note : If $\xi : \mathbb{H} \to \overline{\mathbb{Z}}_p$ is classical arising from $\lambda : \mathbb{H}_k \to \overline{\mathbb{Z}}_p$, then

$$\mathbb{Z}_p[[\Gamma]] \xrightarrow{w} \overbrace{\mathbb{H} \twoheadrightarrow \mathbb{H}_k^{(p)} \otimes \mathbb{Z}_p \xrightarrow{\lambda \otimes id} \overline{\mathbb{Z}}_p}^{\xi}$$

$$\ell \longmapsto S_\ell \qquad\qquad \ell^{k-2}$$

$$x \longmapsto \qquad x^{k-2}$$

$$\prod \mathbb{H}_k \otimes \mathbb{Z}_p \twoheadrightarrow \mathbb{H}_k \otimes \mathbb{Z}_p \xrightarrow{\lambda} \overline{\mathbb{Z}}_p$$

$$(\ell^{k-2})_k \longmapsto \ell^{k-2}$$

$$\kappa_k = \xi \circ w$$

Think of w mapping a system of Hecke eigenvalues to its corresponding weight.

As $\xi \circ w$ is κ_k, it forces w to be injective

$$\mathbb{H} \xrightarrow[\text{classical}]{} \overline{\mathbb{Z}}_p \qquad \mathbb{Z}[[\Gamma]] \to \mathbb{H}$$

$$\left(\text{as } \underset{\mathbb{Z}_p[[\Gamma]]}{\mathbb{Z}_p[[T]]} \xrightarrow{w} \mathbb{H} \longrightarrow \overline{\mathbb{Z}}_p \right.$$

$$T \longmapsto -1 + (1+q)^{k-2}$$

$k \gg 0$ will ensure nothing nonzero dies)

As w is injective, $\operatorname{Spec} \mathbb{H} \longrightarrow \operatorname{Spec} \mathbb{Z}[[\Gamma]]$ is sch. theoretically dominant & $\therefore$ set theoretically.

We can ask if $\exists$ families of systems of Hecke eigenvalues (& $\therefore$ of Galois representations) parametrized by weight.

$$\Gamma_0(N)/\Gamma_1(N) \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^\times$$

$$\alpha \longrightarrow \ell$$

$$\alpha = \begin{pmatrix} 1 & b \\ 0 & \ell \end{pmatrix} \qquad a\ell \equiv 1 \bmod N$$

$$\langle \ell \rangle = \Gamma_1(N)\, \alpha\, \Gamma_1(N) \qquad \langle \ell \rangle, \quad \ell^{k-2} \langle \ell \rangle$$

$$\lambda(S^\ell) = \varepsilon(\ell)\, \ell^{k-2}$$

$$\log_\ell | \quad | = k-2$$

Can we find

$$Z \xhookrightarrow{\text{closed}} \operatorname{Spec} \mathbb{H} \quad \text{s.t.}$$

1) $Z \hookrightarrow \operatorname{Spec} \mathbb{H} \xrightarrow{w} \operatorname{Spec} \mathbb{Z}_p[[T]]$ is dominant with finite fibers

2) Z contains a Zariski dense set of points corresponding to classical systems of Hecke eigenvalues

§3. E.g. of such a family : the Eisenstein family

For simplicity $N = 1$ & fix an even residue class i mod $p-1$ if p is odd.

(Recall : If $k \geq 4$, even, $E_k \in M_k(1)$ is a Hecke eigenform)

Consider $\lambda_k^{(p)}$ λ_k restricted to $\mathbb{H}_k^{(p)}$ associated to Eisenstein series E_k for $k \geq 4$ & $\begin{cases} k \equiv i \mod p-1 & \text{if } p \text{ is odd} \\ k \text{ even} & \text{if } p = 2 \end{cases}$

$$\lambda_k^{(p)}(\ell S_\ell) = \ell^{k-1} \qquad \lambda_k^{(p)}(T_\ell) = 1 + \ell^{k-1}$$

$$\mathbb{Z}_p^\times = \mu \times \Gamma \qquad \left(\begin{array}{ll} \mu = \mu_{p-1} & \text{if } p \text{ is odd} \\ \mu = \mu_2 & \text{if } p \text{ is even} \end{array}\right)$$

Let $\varphi : \mathbb{Z}_p^\times \longrightarrow \mu$

$$\lambda_k^{(p)}(\ell S_\ell) = \ell\, \varphi(\ell)^{i-2} \left(\ell\, \varphi(\ell)^{-1}\right)^{k-2}$$

$$\lambda_k^{(p)}(T_\ell) = 1 + \ell\, \varphi(\ell)^{i-2} \left(\ell\, \varphi(\ell)^{-1}\right)^{k-2}$$

where we set $i = 0$ if $p = 2$

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{E} & \mathbb{Z}_p[[\Gamma]] \\ S_\ell & \longmapsto & \varphi(\ell)^{i-2}\,[\ell\,\varphi(\ell)^{-1}] \\ T_\ell & \longmapsto & 1 + \ell\,\varphi(\ell)^{i-2}\,[\ell\,\varphi(\ell)^{-1}] \end{array}$$

By construction $\kappa_k \circ E = \lambda_k^{(p)}$ for any $k \equiv i \bmod p-1$ (or even k if $p = 2$)

$x \mapsto x^{k-2}$

$\Gamma \to \mathbb{Z}_p^\times$

$$\mathbb{Z}_p[[\Gamma]] \xrightarrow{w} \mathbb{H} \xrightarrow{E} \mathbb{Z}_p[[\Gamma]]$$

$$[\ell] \longmapsto S_\ell \longmapsto \varphi(\ell)^{i-2}\,[\ell\varphi(\ell)^{-1}]$$

$\ell \in \mathcal{L}$

$\ell \equiv 1 \bmod N\mathfrak{q}$

$= [\ell]$ $\uparrow$ $\because$ $\varphi(\ell) = 1$ as $\ell \equiv 1 \bmod \mathfrak{q}$

$\therefore$ $E \circ w = \mathrm{id}$ on $\mathbb{Z}_p[[\Gamma]] \Rightarrow$ we have a section of the weight map, which is a separated

$\therefore$ E gives a closed immersion $\operatorname{Spec} \mathbb{Z}_p[[\Gamma]] \longrightarrow \operatorname{Spec} \mathbb{H}$

$$\operatorname{Spec} \mathbb{Z}_p[[\Gamma]](\overline{\mathbb{Z}}_p) \longrightarrow \operatorname{Spec} \mathbb{H}(\overline{\mathbb{Z}}_p)$$

$$\kappa \longmapsto \kappa \circ E$$

$$\kappa_k \longmapsto \lambda_k^{(p)}$$

Suppose we included information on T_p & pS_p in λ_k

we would want that if κ_k & $\kappa_{k'}$ are "close p-adically", then λ_k & $\lambda_{k'}$ should also be close p-adically

$(\operatorname{Spec} \mathbb{H})(R)$

$\because$ $\operatorname{Hom}_{rg}(\mathbb{H}, R)$ $\subset$ topological rg

endow with the weakest topology s.t. ev_a is continuous for all $a \in \mathbb{H}$

Say $k' > k$, $k' - k = p^m u$

$pS_p \xrightarrow{\lambda_k} p^{k-1}$

$\lambda_{k'}$: $p^{k'-1}$

$$\text{diff} = p^{k-1}(\underbrace{1 - p^{k'-k}}_{\text{unit}})$$

§4. Hida family & the eigencurve.

We observe that $\lambda_k(T_p)$ & $p\lambda_k(S_p)$ do not interpolate well.

If we consider the p^{th} Hecke polynomial $X^2 - \lambda_k(T_p)X + p\lambda_k(S_p)$, in the preceding e.g., it has the form $X^2 - (1+p^{k-1})X + p^{k-1}$

$$= \underbrace{(X-1)}_{\text{No problem}}\ \underbrace{(X-p^{k-1})}_{\text{Problem.}}$$

So we consider points in $\mathrm{Spec}\,\mathbb{H} \times_{\mathbb{Z}_p} \mathbb{G}_m$ $\subset \mathbb{Z}_p[T,T^{-1}]$

Let $\mathcal{X}$ denote $\overline{\mathbb{Q}}_p$ valued pts of $\mathrm{Spec}\,\mathbb{H} \times \mathbb{G}_m$ consisting of pairs (ξ, α) where $\xi : \mathbb{H} \to \overline{\mathbb{Z}}_p$ is classical coming from some $\lambda : \mathbb{H}_k \to \overline{\mathbb{Z}}_p$ & α is a root of p^{th} Hecke polynomial of λ

$$\mathcal{X}^{ord} := \{(\xi, \alpha) \in \mathcal{X} \mid \alpha \in \overline{\mathbb{Z}}_p^{\times}\} = \overline{\mathbb{Z}}_p \text{ points in } \mathcal{X}$$

Last time :

N, p are fixed. $\ell \neq p$, and a prime.

1) Defined p-adic Hecke algebra $= \varprojlim \mathbb{H}^{(p)}_{\leq k} \otimes \mathbb{Z}_p$

2) Constructed a "weight space" $\quad \operatorname{Spec} \mathbb{H} \xrightarrow{w} \operatorname{Spec} \mathbb{Z}_p[[T]]$

3) Considered an e.g. of a family of systems of Hecke eigenvalues parametrized by weight

(in other words, $\quad Z \hookrightarrow \operatorname{Spec} \mathbb{H} \xrightarrow{w} \operatorname{Spec} \mathbb{Z}_p[[T]]$

$\subset$ Zariski dense set of pts corr. to classical systems. $\quad$ dominant w/ finite fibres)

Now : Hida family & the eigencurve

We starting considering points in $\operatorname{Spec} \mathbb{H} \times_{\mathbb{Z}_p} \mathbb{G}_m$ $\quad \mathbb{Z}_p[T, T^{-1}]$

& defined $\mathcal{X} := \{ (\xi, \alpha) \in \operatorname{Spec} \mathbb{H} \times_{\mathbb{Z}_p} \mathbb{G}_m (\overline{\mathbb{Q}}_p) \mid \xi : \mathbb{H} \to \overline{\mathbb{Z}}_p$ is classing coming from some $\lambda : \mathbb{H}_k \to \overline{\mathbb{Z}}_p$ & α is a root of $\underbrace{p^{th} \text{ Hecke polynomial}}_{X^2 - \lambda(T_p) X + p \lambda(S_p)}$ of λ

$\lambda(S_p) \leftarrow p^{k-2} \langle p \rangle$

We defined $\mathcal{X}^{ord} := \{ (\xi, \alpha) \in \mathcal{X} \mid \alpha \in \overline{\mathbb{Z}}_p^\times \} =$ $\overline{\mathbb{Z}}_p$ points in $\mathcal{X}$

<u>Thm (Hida)</u> : Zariski closure $\mathcal{C}^{ord}$ of $\mathcal{X}^{ord}$ in $\operatorname{Spec} \mathbb{H} \times \mathbb{G}_m$ is 1-dim.

$\mathcal{C}^{ord} \hookrightarrow \operatorname{Spec} \mathbb{H} \times \mathbb{G}_m \xrightarrow{pr} \operatorname{Spec} \mathbb{H} \xrightarrow{w} \operatorname{Spec} \mathbb{Z}_p[[\Gamma]]$ is finite. It is étale in the nbhd of those pts of $\mathcal{X}^{ord}$ coming from systems of eigenvalues appearing in wt $k \geq 2$

If we try to interpolate $\mathcal{X}$, taking alg Zariski closure is "too coarse". Instead, we construct a rigid analytic family lying inside the ass. rigid analytic space of $(\operatorname{Spec} \mathbb{H} \times \mathbb{G}_m)^{an}$.

Thm (Coleman & Mazur): The rig. an. Zariski closure C of $\mathcal{X}$ in $(\mathrm{Spec}\,\mathbb{H} \times \mathbb{G}_m)^{an}$ is 1-dim

The composite

$$C \hookrightarrow (\mathrm{Spec}\,\mathbb{H} \times \mathbb{G}_m)^{an} \longrightarrow (\mathrm{Spec}\,\mathbb{H})^{an} \longrightarrow (\mathrm{Spec}(\mathbb{Z}_p[[T]]))^{an}$$

is flat & has discrete fibers

For any $c > 0$, $\exists$ only finitely many pts (ξ, α) in any given fiber with $\mathrm{ordr}_p(\alpha) \leq c$.

The curve C is called the <u>eigencurve</u> of tame level N. $(C^{ord})^{an}$ is called the "slope 0 part" or "the ordinary part"

§5. Very very rough idea of proofs:

Step 1: Space on which $\mathbb{H}$ acts

(i) "generalized p-adic modular functions"

(ii) surrogate of (i) constructed from gp cohomology of $\Gamma_1(N)$

(iii) p-adically completed cohomology of modular curves.

Step 2: (For (i) & (ii))

Introduce U_p operator on the space

Recall: On q-expansions, $f = \sum a_n q^n$, $U_p f = \sum a_{np} q^n$

Let $\mathbb{H}^{\bullet} :=$ quotient of $\mathbb{H}[U_p]$ that acts faithfully on our space

$$\mathbb{H} \times \mathbb{Z}_p[X] \longrightarrow \mathrm{End}(\text{space})$$
$$X \longmapsto U_p$$

$$\operatorname{Spec} \mathbb{H}^* \hookrightarrow \operatorname{Spec} \mathbb{H} \times \mathbb{A}^1$$

If f is a modular form of wt k & level N, $p \nmid N$, & if α & β are roots of $(X^2 - \lambda(T_p)X + p\lambda(S_p))$

Then $f(\tau) - \beta f(p\tau)$ turns out to be a U_p eigenform of level Np, with U_p eigenvalue α.

$\therefore$ (ξ, α) defines a rg hom $\mathbb{H}^* \to \overline{\mathbb{Q}}_p$

(ξ, α) :

$$\begin{tikzcd} \operatorname{Spec} \overline{\mathbb{Q}}_p \arrow[r] \arrow[dr, dashed] & \operatorname{Spec} \mathbb{H} \times \mathbb{G}_m \arrow[r] & \operatorname{Spec} \mathbb{H} \times \mathbb{A}^1 \\ & \operatorname{Spec} \mathbb{H}^* \arrow[ur, hook] & \end{tikzcd}$$

$$\overline{\mathcal{X}} \subset \operatorname{Spec} \mathbb{H}^*$$

But $\operatorname{Spec} \mathbb{H}^*$ is too big

If f is an eigenform for $\mathbb{H}^*$ whose U_p-eigenvalue α is of +ve slope then $U_p^n f = \alpha^n f \to 0$

We can "cut out the ordinary part".

Quotient of $\mathbb{H}^*$ acting faithfully on the ordinary part is the coordinate rg of $\mathcal{C}^{ord} = \overline{\mathcal{X}}^{ord}$

For Coleman & Mazur's curve, the issue is :

Suppose α has +ve slope, then

$$\begin{array}{ccccccc} \mathbb{H} & \longrightarrow & \mathbb{H}[U_p] & \twoheadrightarrow & \mathbb{H}^* & \longrightarrow & \bar{\mathbb{Z}}_p \\ & & U_p & \longmapsto & \cdots & \longmapsto & \alpha \in \text{max ideal} \end{array}$$

$\mathrm{Im}(U_p)$ in $\mathbb{H}^* \in$ maximal ideal $\underline{\mathfrak{m}^*}$

Say $\mathfrak{m}^*$ lies over $\mathfrak{m}$ in $\mathbb{H}$

$$\underbrace{\widehat{\mathbb{H}^*_{\mathfrak{m}^*}}} \cong \underbrace{\widehat{\mathbb{H}_{\mathfrak{m}}}}[[U_p]]$$

completions

$\therefore$ if $\mathbb{H}_{\mathfrak{m}} \to \bar{\mathbb{Z}}_p$ is any system of eigenvalues, can be extended arbitrarily to $\mathbb{H}^*_{\mathfrak{m}^*}$, by assigning a positive slope value to U_p

For non ordinary stuff, no way of algebraically distinguishing positive slope roots of p^{th} Hecke polynomial from any other +ve slope elts of $\bar{\mathbb{Z}}_p$

By passing to "some analytic setting",
we try to get U_p to be a compact operator w/ reasonable spectral theory & analyze its eigenspaces to prove the theorem.

In the interpolation paper

- Not exactly a direct action of U_p.
  Instead all of $GL_2(\mathbb{Q}_p)$ acts

- Introduction of U_p & passage to its eigenspace is effected by applying the Jacquet module functor.