ALLOWING RAMIFICATION AT EXTRA PRIMES

Goal: Examine behavior of derived deformation ring when adding a prime to the set of ramification

Setup:

p prime. k finite of char p.
G split, semisimple adjoint, $/W(k)$. (eg. PGL_n)

(In particular reductive with trivial center)

$T \subset G$ max k-split torus

$\mathfrak{g} = \mathrm{Lie}(G_k)$

$S =$ finite set of primes containing p

$$\rho : \underset{\pi_1^{\acute{e}t}(\mathrm{Spec}\,\mathbb{Z}[\frac{1}{S}])}{\overset{\shortparallel}{\Gamma_S}} \longrightarrow G(k)$$

(eventually with some conditions to make centralizer $= Z_G$, s.t. representable.)

$\mathcal{F}_S$ the deformation functor lifting ρ.

q: Taylor Wiles prime i.e.

- $q \notin S$
- $q \equiv 1$ in k
- $\rho(\mathrm{Frob}_q)$ is conjugate to a strongly regular element $t \in T(k)$ s.t. $Z_G(t) = T$

(e.g. in PGL_n, distinct e.v.)

$\therefore$ we may fix $\rho^T_{\mathbb{Q}_q}$:

$$\begin{tikzcd}
\pi_1 \mathbb{Q}_q \arrow[r, "\rho^T_{\mathbb{Q}_q}"] \arrow[d] & T(k) \\
\pi_1 \mathbb{Z}_q \arrow[ur, dotted, "\rho^T_{\mathbb{Z}_q}"'] &
\end{tikzcd}$$

such that $inc^G_T \circ \rho^T_{\mathbb{Z}_p} \cong \rho_{\mathbb{Z}_q}$

$inc^G_T \circ \rho^T_{\mathbb{Q}_p} \cong \rho_{\mathbb{Q}_q}$

- $\mathcal{F}_{\mathbb{Z}_q}$, $\mathcal{F}_{\mathbb{Q}_q}$: Deformation functors for $\rho_{\mathbb{Z}_q}$ & $\rho_{\mathbb{Q}_q}$ as reps into G.

- $\mathcal{F}^T_{\mathbb{Z}_q}$, $\mathcal{F}^T_{\mathbb{Q}_q}$: .... $\rho^T_{\mathbb{Z}_q}$ & $\rho^T_{\mathbb{Q}_q}$ valued in T.

Choice of isom above induces:
$\mathcal{F}^T_{\mathbb{Z}_q} \longrightarrow \mathcal{F}_{\mathbb{Z}_q}$, similar for $\mathbb{Q}_q$.

- $\mathcal{F}^{T,\square}_{\mathbb{Z}_q}$, $\mathcal{F}^{T,\square}_{\mathbb{Q}_q}$: framed deformation functors for $\rho^T_{\mathbb{Z}_q}$ & $\rho^T_{\mathbb{Q}_q}$.

- S°_q : Underived framed deformation rg for the trivial rep:
$I_q \longrightarrow T$

Main result:

$$\begin{tikzcd}
\mathcal{F}_{\mathbb{Z}[\frac{1}{S}]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] \arrow[dr, phantom, "(A)"] & \mathcal{F}_{\mathbb{Z}_q} \arrow[d] \arrow[dr, phantom, "(B)"] & \mathcal{F}^T_{\mathbb{Z}_q} \arrow[l, "\sim"'] \arrow[d] \arrow[r, bend left, "s_{\mathbb{Z}_q}"] \arrow[dr, phantom, "(C)"] & \mathcal{F}^{T,\square}_{\mathbb{Z}_q} \arrow[l] \arrow[r] \arrow[d] \arrow[dl, phantom, "\llcorner" very near start] \arrow[dr, phantom, "\lrcorner" very near start] \arrow[dr, phantom, "(D)"] & * \arrow[d] \\
\mathcal{F}_{\mathbb{Z}[\frac{1}{Sq}]} \arrow[r] & \mathcal{F}_{\mathbb{Q}_q} & \mathcal{F}^T_{\mathbb{Q}_q} \arrow[l, "\sim"] \arrow[r, bend right, "s_{\mathbb{Q}_q}"'] & \mathcal{F}^{T,\square}_{\mathbb{Q}_q} \arrow[l] \arrow[r] & \mathrm{Hom}_{\mathrm{Ring}^{sm}_{/R}}(S^\circ_q, -)
\end{tikzcd}$$

B, C, D need that q is a Taylor Wiles prime

((A) has the important idea we need, but we travel through the squares because $\mathcal{F}_{\mathbb{Z}_q}$ & $\mathcal{F}_{\mathbb{Q}_q}$ are not representable))

(A)

$$\begin{tikzcd} \mathcal{F}_{\mathbb{Z}[\frac{1}{S}]} \arrow[r, "f_1"] \arrow[d, "f_2"'] & \mathcal{F}_{\mathbb{Z}_q} \arrow[d] \\ \mathcal{F}_{\mathbb{Z}[\frac{1}{Sq}]} \arrow[r] & \mathcal{F}_{\mathbb{Q}_q} \end{tikzcd}$$

Want the natural map to be a w.h.e $\mathcal{F}_{\mathbb{Z}[\frac{1}{S}]} \xrightarrow{\sim} \mathcal{F}_{\mathbb{Z}_q} \times_{\mathcal{F}_{\mathbb{Q}_q}} \mathcal{F}_{\mathbb{Z}[\frac{1}{Sq}]}$

Suffices to check on $k \oplus k[m]$ $\forall m \geq 0$ because

1) If $A \in \mathrm{Ring}^{sm}_{/k}$, $\varepsilon: A \to k$ factors as

$$A = A_0 \to A_1 \to A_2 \to \cdots \to A_n = k$$

where each A_i is a square-zero ext of A_{i+1} by $k[m_i]$ $m_i \geq 0$

2) $\mathcal{F}_{..}$ preserves pullbacks by small extensions (checked last time) and therefore so does $\mathcal{F}_{\mathbb{Z}_q} \times_{\mathcal{F}_{\mathbb{Q}_q}} \mathcal{F}_{\mathbb{Z}[\frac{1}{Sq}]}$

Therefore we need to show:

$$t\mathcal{F}_{\mathbb{Z}[\frac{1}{S}]} \xrightarrow{\sim} t\mathcal{F}_{\mathbb{Z}_q} \times_{t\mathcal{F}_{\mathbb{Q}_q}} t\mathcal{F}_{\mathbb{Z}[\frac{1}{Sq}]}$$

((Since Mod_k is an additive category & pullbacks are the same as pushouts))

Equivalently, that

$$\rightsquigarrow \quad t\mathcal{F}_{\mathbb{Z}[\frac{1}{S}]} \xrightarrow{tf_1, -tf_2} t\mathcal{F}_{\mathbb{Z}_q} \oplus t\mathcal{F}_{\mathbb{Z}[1/Sq]} \longrightarrow t\mathcal{F}_{\mathbb{Q}_q}$$

is a fiber sequence

By explicit calculations of these tangent complexes by Shenrong:

NTS:

$$\begin{tikzcd} C^*(\mathbb{Z}[\frac{1}{S}], \rho^*\mathfrak{g}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & C^*(\mathbb{Z}_q, \rho^*_{\mathbb{Z}_q}\mathfrak{g}) \oplus C^*(\mathbb{Z}[\frac{1}{Sq}], \rho^*_{..}\mathfrak{g}) \arrow[d] \\ 0 \arrow[r] & C^*(\mathbb{Q}_q, \rho^*_{\mathbb{Q}_q}\mathfrak{g}) \end{tikzcd}$$

Note: These complexes compute étale cohomologies.

((Probably true by definition. Constructed by taking limit of $\mathrm{Simp}(X) \xrightarrow{p^* tBG} \mathrm{Ch}(k)$ where X is $\{X_\alpha\}_\alpha$ pro-simplicial, with each X_α is an étale hypercovering of Spec ...))

↳ adding more than the Čech nerve of an étale cover of Spec...

To verify this:

Replace $\mathrm{Spec}\, \mathbb{Z}_q$ by $\mathbb{Z}_q^{hs}$, the henselization of $\mathbb{Z}$ at the closed pt q.

We have $\mathbb{Z}_{(q)} \subset \mathbb{Z}_q^{hs} \subset \mathbb{Z}_q$

integral

henselian local rings with same residue field

$\Rightarrow$ isomorphic category of finite étale covers, same as those of $\mathbb{F}_q$

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finiteness suffices probably because we have finite coeffs.

$\Rightarrow$ can replace $C^*(\mathbb{Z}_q)$ with $C^*(\mathbb{Z}_q^{hs})$

& $C^*(\mathbb{Q}_q)$ with $C^*(\mathbb{Q}_q^{hs})$

$\mathbb{Z}_q^{hs}[\frac{1}{q}]$

Henselization can be presented as: 044V

$$\varinjlim_V \mathcal{O}_V(V)$$

$$\begin{tikzcd} & V \arrow[d, "\text{étale cover}"] \\ \mathrm{Spec}\, \mathbb{F}_q \arrow[ur, dashed] \arrow[r] & \mathrm{Spec}\, \mathbb{Z}[\frac{1}{s}] \end{tikzcd}$$

$$\Rightarrow \varinjlim C^*_{\acute{e}t}(V) \xrightarrow{\sim} C^*_{\acute{e}t}(\mathbb{Z}_q^{hs})$$

A similar statement ends up being true for $\mathbb{Q}_q$

$$\varinjlim C^*_{\acute{e}t}(V \times_{\mathbb{Z}[\frac{1}{s}]} \mathbb{Z}[\tfrac{1}{sq}]) \xrightarrow{\sim} C^*_{\acute{e}t}(\mathbb{Q}_q^{hs})$$

For $V \xrightarrow[\text{cover}]{\text{étale}} \mathrm{Spec}\, \mathbb{Z}[\frac{1}{s}]$, $\mathcal{U} = \mathrm{Spec}\, \mathbb{Z}[\frac{1}{sq}] \underset{\text{open}}{\subseteq} \mathrm{Spec}\, \mathbb{Z}[\frac{1}{s}]$,

$$C^*(\mathrm{Spec}\, \mathbb{Z}[\tfrac{1}{s}]) \longrightarrow C^*(\mathcal{U}) \oplus C^*(V) \longrightarrow C^*(V \times_{\mathbb{Z}[\frac{1}{s}]} \mathcal{U})$$

is a fiber sequence (Mayer Vietoris 0A50)

Can check by passing to injective resolution.

(B)

want to show this & bottom row are weak equiv

$$\begin{tikzcd} \mathcal{J}^T_{\mathbb{Z}_q} \arrow[r] \arrow[d] & \mathcal{J}_{\mathbb{Z}_q} \arrow[d] \\ \mathcal{J}^T_{\mathbb{Q}_q} \arrow[r] & \mathcal{J}_{\mathbb{Q}_q} \end{tikzcd}$$

$\rightsquigarrow$ tangent complexes

$\rightsquigarrow$ homotopy groups

$$\begin{tikzcd} H^*_{et}(\mathbb{Z}_q, \mathrm{Lie}(T)_k) \arrow[r] \arrow[d] & H^*_{et}(\mathbb{Z}_q, \mathfrak{g}) \arrow[d] \\ H^*_{et}(\mathbb{Q}_q, \mathrm{Lie}(T)_k) \arrow[r] & H^*_{et}(\mathbb{Q}_q, \mathfrak{g}) \end{tikzcd}$$

Want to show that top & bottom arrows are isoms.

First, note that as $\mathrm{spec}\, \mathbb{Z}_q$ is Henselian,
$H^i_{et}(\mathrm{Spec}\, \mathbb{Z}_q) = H^i_{et}(\mathbb{F}_q)$ which is the Galois coh. for $\Gamma^{ur}_{\mathbb{Q}_p}$

Stacks 03QU

$H^i_{et}(\mathrm{Spec}\, \mathbb{Q}_q)$ is the Galois coh of $G_{\mathbb{Q}_q}$

$\therefore \; h^i(\mathbb{Z}_q) = 0 \quad \forall i > 1$

$h^i(\mathbb{Q}_q) = 0 \quad \forall i > 2$

H^0: $\mathfrak{g}^{\Gamma_{\mathbb{Q}_p}} = \mathfrak{g}^{\Gamma^{ur}_{\mathbb{Q}_p}} = \mathfrak{g}^{Ad(\rho(\mathrm{Frob}_q))} = \mathrm{Lie}(Z_G(t))_k = \mathrm{Lie}(T)_k$

$\rho(\mathrm{Frob}_q)$ is strongly regular

H^2: bottom map:

Local Tate duality gives that

$H^2(\Gamma_{\mathbb{Q}_q}, M) \cong H^0(\Gamma_{\mathbb{Q}_q}, M^*)^\vee$

Therefore STS isom on H^0

$M^* = \mathrm{Hom}_{gp}(M, \mu_p) = \mathrm{Hom}(M, \mathbb{Z}/p\mathbb{Z}) = M^\vee$

p torsion

As $q \equiv 1$ in k

$\Rightarrow H^0(\Gamma_{\mathbb{Q}_q}, \mathfrak{g}^\vee) = (\mathrm{Lie}\, T)^\vee$

$\| $

dual of coinv of $\mathfrak{g}$

$\|$

$\mathrm{Lie}(T)$

$\because \mathfrak{g}_\alpha$ has action by $s(a) \neq 1$

$= H^0(\Gamma_{\mathbb{Q}_q}, (\mathrm{Lie}\, T)^\vee)$

$(s(a)-1)X \in$ kernel

$\Rightarrow (s(a)-1)^{p^r-1} X \in$ kernel

$\Rightarrow X \in$ kernel

$k = \mathbb{F}_{p^r}$

$\Rightarrow h^2(\mathbb{Q}_q, \mathfrak{g}) = \dim(\mathrm{Lie}\, T) = \mathrm{rank}\, G$

H^1 : As $\mathrm{Lie}(T)_k$ is a summand of $\mathfrak{g}$, necessarily inj.

bottom row :

By Euler char. formula :

$$\underset{\text{as } p \neq q}{1 =} \chi(\Gamma_{\mathbb{Q}_q}, M) = \frac{\# H^0(\mathbb{Q}_q, M) \cdot \# H^2(\mathbb{Q}_q, M)}{\# H^1(\mathbb{Q}_q, M)}$$

$\Rightarrow H^1(\mathbb{Q}_p, \mathrm{Lie}(T)_k) \longrightarrow H^1(\mathbb{Q}_p, \mathfrak{g})$ is an injection of finite gps of same size $\Rightarrow$ isom.

top row :

isom on $(\mathrm{Lie}\, T)_k$ summand of $\mathfrak{g}$, therefore suffices to check for nonzero root spaces of $\mathfrak{g}$

(((each is 1-dim & preserved under $\Gamma_{\mathbb{Q}_q}$ action because $\Gamma_{\mathbb{Q}_q}$ maps into T)))

So we may consider :

$H^1(\hat{\mathbb{Z}}, k)$ where $\hat{\mathbb{Z}}$ acts nontriv on k via character α

$$\cong \frac{k}{(\alpha(1) - 1)k} = 0$$

Corollary :

$\rho^T_{\mathbb{Q}_q} \xrightarrow{\sim} \rho_{\mathbb{Q}_q}$ is an equivalence

$\Rightarrow$ for any lift of $\rho_{\mathbb{Q}}$, we have a conjugati T-valued lift

$\Rightarrow$ $\Gamma_{\mathbb{Q}_q}$ action factors through $\pi_1 \mathbb{Q}_q^{tame, ab}$

(abelian $\because$ T is abelian

tame $\because$ I_q maps into pro-p gp whereas wild I_q is pro-ℓ)

(C)

$$\begin{tikzcd}
\mathcal{F}^T_{\mathbb{Z}_q} \arrow[d] & \mathcal{F}^{T,\square}_{\mathbb{Z}_q} \arrow[l] \arrow[d] \\
\mathcal{F}^T_{\mathbb{Q}_q} & \mathcal{F}^{T,\square}_{\mathbb{Q}_q} \arrow[l]
\end{tikzcd}$$

Pullback because $\mathcal{F}^{T,\square}_{\mathbb{Q}_q}(A) = \mathcal{F}^T_q(A) \times_{BG(A)} *$ $\leftarrow$ basept of $BG(A)$

(evaluation at basepoint of X

$\mathcal{F}^T_q(A) = \mathrm{fib}(\mathrm{Hom}(X, BG(A)) \to \rho \in \mathrm{Hom}(X, BG(k)))$)

Now we construct splittings

$$\begin{tikzcd}
\mathcal{F}^T_{\mathbb{Z}_q} \arrow[r, "s_{\mathbb{Z}_q}"] \arrow[d] \arrow[rr, dashed, bend left, "\mathrm{id}"] \arrow[dr, phantom, "(C')"] & \mathcal{F}^{T,\square}_{\mathbb{Z}_q} \arrow[r] \arrow[d] \arrow[dr, phantom, "(C)"] & \mathcal{F}^T_{\mathbb{Z}_q} \arrow[d] \\
\mathcal{F}^T_{\mathbb{Q}_q} \arrow[r, "s_{\mathbb{Q}_q}"'] \arrow[rr, dashed, bend right, "\mathrm{id}"'] & \mathcal{F}^{T,\square}_{\mathbb{Q}_q} \arrow[r] & \mathcal{F}^T_{\mathbb{Q}_q}
\end{tikzcd}$$

These would show that (C') is a pullback square

To make the splittings, we note that the commutativity of T makes $BT(A)$ a simplicial group.

Then

$$\begin{tikzcd}
\mathrm{Hom}_{sSet}((X,*), (BT(A),*)) \arrow[r] \arrow[d] & \mathrm{Hom}_{sSet}(X, BT(A)) \arrow[l, bend right, dashed] \arrow[d] \\
\mathrm{Hom}_{sSet}((X,*), (BT(k),*)) \arrow[r] & \mathrm{Hom}_{sSet}(X, BT(k)) \arrow[l, bend left, dashed]
\end{tikzcd}$$

use gp structure on $BT(A)$ to send basepoint to basepoint

This gives splittings on $\underbrace{\text{fibers over } \rho}_{\text{deformation functors of } \rho}$.

(D)

Lemma: Suppose $R \in \mathrm{Ring}^{cq}_{/k}$ $b_i = \dim(t^i R)$ & $b_i = 0$ for $i > 1$ (& also < 0 probably automatic)

$\hookrightarrow \pi_0(T_{X,x})_i$

(Defined by David last semester. these were the representing objects in derived Schlessinger's)

(Def: $\mathrm{Ring}^{cq}_{/k} \hookrightarrow \mathrm{Ring}_{/k}$ is full on the $R \xrightarrow{\varepsilon} k$ s.t. (i) R (is connective) & noeth

↳ i.e. $\pi_0 R$ is noeth & each $\pi_n R$ is f.g. $\pi_0 R$-mod

(ii) $\pi_0 R \to k$ is surjective

(iii) $\pi_0 R$ is local & complete w.r.t. m w/ residue field k.)

Then R is discrete $\iff$ $\pi_0 R = \dfrac{W(k)[[x_1, \ldots, x_{b_0}]]}{y_1, \ldots, y_{b_1}}$ for a regular seq y_i of elements in (p, m^2).

Pf: Skipped

Maybe Ashwin's talk in FRG seminar last Semester.

Lemma: The representing rings for $\mathcal{F}^{T,\square}_{\mathbb{Q}_q}$ & $\mathcal{F}^{T,\square}_{\mathbb{Z}_q}$ are discrete.

Pf: Let $S_q^{ur} = \pi_0$ (Representing ring for $\mathcal{F}^{T,\square}_{\mathbb{Z}_q}$)

$t^{\circ}R = S_q^{ur}(k[\varepsilon]/\varepsilon^2)$
$= k^{\oplus r}$

$S_q = \pi_0$ (" " " $\mathcal{F}^{T,\square}_{\mathbb{Q}_q}$)

Note that these are the underived deformation rings.

((π_0 is the left adjoint to inclusion of classical rings))

Let $\dim T = r$

S_q^{ur} is a power series ring as we just need to specify lift of $\rho(\mathrm{Frob}_q)$.

$$S_q^{ur} = W(k)[[X_1, \ldots, X_r]]$$

$t^{\circ} = S_q(k[\varepsilon]/\varepsilon^2)$
$k^{\oplus r \oplus r}$

Now, S_q.

Any (underived) T-valued deformation of $\rho^T_{\mathbb{Q}_q}$ factors through the tame abelian quotient of $\pi_1 \mathbb{Q}_q$.

We have $\pi_1 \mathbb{Q}_q^{tame, ab} \cong \langle \mathrm{Frob}_q \rangle \times I_q$

non-canonical

cyclic of order $q-1$

Suppose N is max s.t. $p^N \mid q-1$.

$((r = \dim T))$

$$S_q \cong W(k)[[X_1, \dots, X_r, Y_1, \dots, Y_r]] / \langle (1+Y_i)^{p^N} - 1 \rangle$$

((deformation of matrix of $\rho(\mathrm{Frob}_q)$ — deformation of 1))

satisfies $*$ ✓

$$\begin{tikzcd}
\mathcal{F}^{T,\square}_{\mathbb{Z}_q} \arrow[r] \arrow[d] \arrow[dr, phantom, "(D)"] & * \arrow[d] \\
\mathcal{F}^{T,\square}_{\mathbb{Q}_q} \arrow[r] & \mathrm{Hom}_{\mathrm{Ring}^{sm}/k}(S^\circ_q, -)
\end{tikzcd}$$

$$\begin{tikzcd}
I_q \arrow[r] & \pi'_1 \mathbb{Q}_p^{tame,ab} \arrow[r] & \pi_1 \mathbb{F}_q
\end{tikzcd}$$

induces
$$\begin{tikzcd}
S^\circ_q \arrow[r] & S_q \arrow[r] & S^{ur}_q
\end{tikzcd}$$

framed deformation ring of triv T-valued I_q-rep

Explicitly :

$$\frac{W(k)[[Y_1, \dots, Y_r]]}{\langle (1+Y_i)^{p^N} - 1 \rangle} \longrightarrow \frac{W(k)[[X_1, \dots, X_r, Y_1, \dots, Y_r]]}{(1+Y_i)^{p^N} - 1} \longrightarrow W(k)[[X_i]]$$

$$\begin{tikzcd}
S^\circ_q \arrow[r] \arrow[d] & S_q \arrow[d] \\
W(k) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & S^{ur}_q
\end{tikzcd}$$

induces

$$\begin{tikzcd}
\mathcal{F}^{T,\square}_{\mathbb{Z}_q} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & * \arrow[d] \\
\mathcal{F}^{T,\square}_{\mathbb{Q}_q} \arrow[r] & \mathrm{Hom}_{\mathrm{Rg}/k}(S^\circ_q, -)
\end{tikzcd}$$

✓