CODIMENSION ONE INTERSECTIONS BETWEEN COMPONENTS OF A MODULI STACK OF TWO-DIMENSIONAL GALOIS REPRESENTATIONS

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Abstract. The Emerton-Gee stack for GL_2 is a stack of (φ, Γ)-modules whose reduced part $\mathcal{X}_{2,\mathrm{red}}$ can be viewed as a moduli stack of mod p representations of a p-adic Galois group. We compute criteria for intersections of irreducible components of $\mathcal{X}_{2,\mathrm{red}}$ in codimension 1 and relate them to extensions of Serre weights.

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1. Introduction

Let p be an odd prime and let $K/\mathbb{Q}_p$ be a finite extension, with ring of integers $\mathcal{O}_K$, residue field k and absolute Galois group G_K. In [EG1], Emerton and Gee constructed and studied the stack of rank d étale (φ, Γ)-modules, denoted $\mathcal{X}_d$. Over Artinian coefficients, there exists an equivalence of categories between rank d étale (φ, Γ)-modules and d-dimensional G_K-representations that allows one to view $\mathcal{X}_d$ as a moduli stack of Galois representations. The Emerton-Gee stack $\mathcal{X}_d$ is expected to play a central role in the p-adic Langlands program, occupying the position played by the moduli stack of L-parameters in the work of Fargues-Scholze on the classical Langlands correspondence.

Specializing to $d = 2$, the reduced part of $\mathcal{X}_2$, denoted $\mathcal{X}_{2,\mathrm{red}}$, is an algebraic stack defined over a finite field $\mathbb{F}$. The irreducible components of $\mathcal{X}_{2,\mathrm{red}}$ are labelled by Serre weights, which are the irreducible mod p representations of $\mathrm{GL}_2(k)$. By [CEGS, Cor. 7.2], the labelling is in such a manner that if $\mathcal{X}_{2,\mathrm{red},\sigma}$ is the component labelled by σ, then its finite type points are precisely those representations that have σ as a Serre weight, that is, they have crystalline lifts of Hodge-Tate weights specified in a particular way by σ (see Section 1.4 for details).

The main objective of this article is to compute criteria for pairs of Serre weights σ and τ so that $\mathcal{X}_{2,\mathrm{red},\sigma} \cap \mathcal{X}_{2,\mathrm{red},\tau}$ is a substack of codimension 1. Our strategy rests on finding families of representations that have both σ and τ as Serre weights, and therefore give points of $\mathcal{X}_{2,\mathrm{red},\sigma} \cap \mathcal{X}_{2,\mathrm{red},\tau}$. The sizes of these families can then be used to determine the dimension of $\mathcal{X}_{2,\mathrm{red},\sigma} \cap \mathcal{X}_{2,\mathrm{red},\tau}$. As employed in [EG1], a source of families of representations is provided by extensions of fixed G_K characters together with extensions of their unramified twists. Every irreducible component of $\mathcal{X}_{2,\mathrm{red}}$ can be obtained as the closure of such a family. Vector spaces of extensions of fixed G_K characters are typically $[K : \mathbb{Q}_p]$-dimensional. Allowing various unramified twists of the fixed characters adds 2 to the dimension, while 1 dimension is taken away because a $\mathbb{G}_m$ orbit of an extension class gives the same representation and yet another dimension is taken away because of a $\mathbb{G}_m$ worth of endomorphisms of each extension. Thus a codimension 1 intersection of $\mathcal{X}_{2,\mathrm{red},\sigma}$ and $\mathcal{X}_{2,\mathrm{red},\tau}$ may be expected to correspond to the existence of a codimension 1 subfamily of extensions of fixed G_K characters (as well as their unramified twists) with both σ and τ as their Serre weights. This line of investigation gives us the required criteria (stated in Theorem 7.1). For pairs of non-isomorphic and weakly regular (a very mild genericity hypothesis, see definition in Section 1.4) Serre weights, the criteria are summarized below. The precise criteria for intersections involving components labelled by Serre weights that are not weakly regular are significantly less succinct and omitted from the statement below.

Theorem 1.1. *If σ and τ are a pair of non-isomorphic Serre weights, then*

$$\mathrm{Ext}^1_{\mathbb{F}[\mathrm{GL}_2(\mathcal{O}_K)]}(\sigma, \tau) \neq 0 \implies \dim \mathcal{X}_{2,\mathrm{red},\sigma} \cap \mathcal{X}_{2,\mathrm{red},\tau} = [K : \mathbb{Q}_p] - 1.$$

When σ and τ are also weakly regular, the following stronger statement is true:

$$\mathrm{Ext}^1_{\mathbb{F}[\mathrm{GL}_2(\mathcal{O}_K)]}(\sigma, \tau) \neq 0 \iff \dim \mathcal{X}_{2,\mathrm{red},\sigma} \cap \mathcal{X}_{2,\mathrm{red},\tau} = [K : \mathbb{Q}_p] - 1.$$

This result can be motivated in terms of the conjectural categorical p-adic Langlands correspondence. Specifically, it has been conjectured ([EGH, Conj. 6.1.6]) that there exists a fully faithful functor $\mathfrak{A}$ from a derived category of smooth representations of $\mathrm{GL}_2(K)$ to a derived category of coherent sheaves on $\mathcal{X}_2$ that witnesses the p-adic local Langlands. The functor $\mathfrak{A}$ is expected to satisfy properties related to duality and support that imply the following:

- For σ a non-Steinberg Serre weight, the support of $\mathfrak{A}(\text{c-Ind}_{\mathrm{GL}_2(\mathcal{O}_K)}^{\mathrm{GL}_2(K)}\sigma)$ is $\mathcal{X}_{2,\mathrm{red},\sigma}$.
- For σ, τ Serre weights and $V \in \mathrm{Ext}^1_{\mathbb{F}[\mathrm{GL}_2(\mathcal{O}_K)]}(\sigma, \tau)$, $\mathfrak{A} \circ \text{c-Ind}_{\mathrm{GL}_2(\mathcal{O}_K)}^{\mathrm{GL}_2(K)}(\tau \to V \to \sigma)$ is a short exact sequence.

Therefore,

$$\mathfrak{A} \circ \text{c-Ind}_{\mathrm{GL}_2(\mathcal{O}_K)}^{\mathrm{GL}_2(K)}(V)|_{(\mathcal{X}_{2,\mathrm{red},\sigma} \cap \mathcal{X}_{2,\mathrm{red},\tau})^c} \cong \mathfrak{A} \circ \text{c-Ind}_{\mathrm{GL}_2(\mathcal{O}_K)}^{\mathrm{GL}_2(K)}(\sigma \oplus \tau)|_{(\mathcal{X}_{2,\mathrm{red},\sigma} \cap \mathcal{X}_{2,\mathrm{red},\tau})^c}.$$

Since $\mathfrak{A}$ is fully faithful, if the intersection of $\mathcal{X}_{2,\mathrm{red},\sigma}$ and $\mathcal{X}_{2,\mathrm{red},\tau}$ is empty, then $\text{c-Ind}_{\mathrm{GL}_2(\mathcal{O}_K)}^{\mathrm{GL}_2(K)}(V)$ must be isomorphic to $\text{c-Ind}_{\mathrm{GL}_2(\mathcal{O}_K)}^{\mathrm{GL}_2(K)}\sigma \oplus \text{c-Ind}_{\mathrm{GL}_2(\mathcal{O}_K)}^{\mathrm{GL}_2(K)}\tau$. Thus, we obtain the following diagram of $\mathrm{GL}_2(\mathcal{O}_K)$ representations where the right downward arrow splits:

$$\begin{array}{ccc} \sigma & \longrightarrow & \text{c-Ind}_{\mathrm{GL}_2(\mathcal{O}_K)}^{\mathrm{GL}_2(K)}\sigma \\ \big\downarrow & & \big\downarrow \; \big\uparrow \\ V & \longrightarrow & \text{c-Ind}_{\mathrm{GL}_2(\mathcal{O}_K)}^{\mathrm{GL}_2(K)}V \end{array}$$

The horizontal arrows split as maps of $\mathrm{GL}_2(\mathcal{O}_K)$ representations by Mackey's decomposition theorem. The left vertical arrow must then split as well, and V must be isomorphic to $\sigma \oplus \tau$. This shows that if the conjectured functor $\mathfrak{U}$ exists, then an empty intersection of $\mathcal{X}_{2,\mathrm{red},\sigma}$ with $\mathcal{X}_{2,\mathrm{red},\tau}$ implies that there are no non-trivial extensions of τ by σ as $\mathrm{GL}_2(\mathcal{O}_K)$ modules. Our theorem is a finer variant of this expectation.

In the course of our computations, we also find a cohomological criterion for the number of components of dimension $[K : \mathbb{Q}_p] - 1$ when $\mathcal{X}_{2,\mathrm{red},\sigma} \cap \mathcal{X}_{2,\mathrm{red},\tau}$ is codimension 1, along with some naturally occurring triples of Serre weights. The theorem below summarizes the results for pairs of weakly regular Serre weights σ and τ.

Theorem 1.2. *Let σ and τ be two weakly regular Serre weights such that $\mathcal{X}_{2,\mathrm{red},\sigma} \cap \mathcal{X}_{2,\mathrm{red},\tau}$ is of codimension 1. Then the following are true:*

- *(i) When K is unramified over $\mathbb{Q}_p$, the number of components of dimension $[K : \mathbb{Q}_p] - 1$ in $\mathcal{X}_{2,\mathrm{red},\sigma} \cap \mathcal{X}_{2,\mathrm{red},\tau}$ is 1. When K is ramified over $\mathbb{Q}_p$, this number is 2 if the $\mathrm{GL}_2(k)$-extensions of τ by σ are non-trivial, and 1 otherwise.*
- *(ii) When K is unramified over $\mathbb{Q}_p$, a component of dimension $[K : \mathbb{Q}_p] - 1$ in $\mathcal{X}_{2,\mathrm{red},\sigma} \cap \mathcal{X}_{2,\mathrm{red},\tau}$ does not lie in an intersection of three irreducible components of $\mathcal{X}$. In the ramified case, for sufficiently generic Serre weights (c.f. Theorem 7.3), each component of dimension $[K : \mathbb{Q}_p] - 1$ in $\mathcal{X}_{2,\mathrm{red},\sigma} \cap \mathcal{X}_{2,\mathrm{red},\tau}$ lies in an intersection of three irreducible components of $\mathcal{X}_{2,\mathrm{red}}$.*

Note that the criterion that appears in Theorem 1.1 has to do with $\mathrm{GL}_2(\mathcal{O}_K)$-extensions, while the criterion that appears in Theorem 1.2 has to do with $\mathrm{GL}_2(k)$-extensions.

1.3. **Outline of the paper.** In Section 2, we compute explicit criteria for the existence of non-trivial extensions of Serre weights as $\mathrm{GL}_2(\mathcal{O}_K)$ representations. In Section 3, we relate the dimensions of families of G_K-representations with both σ and τ as Serre weights to the dimension of $\mathcal{X}_{2,\mathrm{red},\sigma} \cap \mathcal{X}_{2,\mathrm{red},\tau}$. We also relate the number of sufficiently large families to the number of components of maximal dimension inside $\mathcal{X}_{2,\mathrm{red},\sigma} \cap \mathcal{X}_{2,\mathrm{red},\tau}$. Section 4 recalls explicit criteria for computations of Serre weights of representations. Along with the results of Section 3, these criteria are used to restructure the problem as that of finding σ and τ that satisfy a precise computable relationship. Sections 5 and 6 compute the solution to the problem laid out in Section 4. Finally, Section 7 collates all the findings.

1.4. **Notation.** Let p be a fixed prime and let K be a finite extension of $\mathbb{Q}_p$ with valuation ring $\mathcal{O}_K$, residue field k and uniformizer π. Eventually, p will be an odd prime, to allow the key input of [CEGS, Cor. 7.2]. However, we will allow $p = 2$ for many of the intermediate steps.

We let $f := f(K/\mathbb{Q}_p)$ and $e := e(K/\mathbb{Q}_p)$. Let G_K be the absolute Galois group of K, and I_K the inertia group. $\mathbb{F}$ is a finite extension of $\mathbb{F}_p$, with a fixed algebraic closure $\overline{\mathbb{F}}$. $\mathbb{F}$ is taken to be sufficiently large so that all embeddings of k into $\overline{\mathbb{F}}$ are contained in $\mathbb{F}$.

Let $T := [0, f-1]$. Fix an embedding $\sigma_{f-1} : k \to \overline{\mathbb{F}}$. Let $\sigma_{f-1-i} := \sigma_{f-1}^{p^i}$ for $i \in T$. Let ω_i be the G_K character given by $\omega_i(g) = \sigma_i(\frac{g(\sqrt[p^f-1]{\pi})}{\sqrt[p^f-1]{\pi}})$.

We let $V_{\vec{t},\vec{s}}$ denote the irreducible $\mathrm{GL}_2(k)$ representation

$$\bigotimes_{i=0}^{f-1}(\det{}^{t_i} \otimes \mathrm{Sym}^{s_i} k^2) \otimes_{k,\sigma_i} \overline{\mathbb{F}}$$

where each $s_i \in [0, p-1]$. All irreducible $\mathrm{GL}_2(k)$ representations with coefficients in $\mathbb{F}$ are of this form and are called Serre weights. We can uniquely identify each Serre weight by $\vec{s}$ and $\vec{t}$ if we demand that $t_i \in [0, p-1]$ $\forall i$ and at least one of the t_i's is not $p-1$. Following [Gee], we say $V_{\vec{t},\vec{s}}$ is *weakly regular*, if each $s_i \in [0, p-2]$. We say that $V_{\vec{t},\vec{s}}$ is Steinberg if each s_i equals $p-1$.

Normalize Hodge-Tate weights in such a way that all Hodge-Tate weights of the cyclotomic character are equal to -1. Consistent with the conventions in [EG1], we say that a representation $\overline{\rho} : \mathrm{G}_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ has Serre weight $V_{\vec{t},\vec{s}}$ if $\overline{\rho}$ has a crystalline lift $\rho : \mathrm{G}_K \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ that satisfies the following condition: For each embedding $\sigma_i : k \hookrightarrow \mathbb{F}$, there is an embedding $\tilde{\sigma}_i : K \hookrightarrow \mathbb{Q}_p$ lifting σ_i such that the $\tilde{\sigma}_i$ labeled Hodge-Tate weights of r are $\{t_i, s_i + t_i + 1\}$, and the remaining $(e-1)f$ pairs of Hodge-Tate weights of r are all $\{0, 1\}$. In this situation, we say $V_{\vec{t},\vec{s}} \in W(\overline{\rho})$.

Let $\mathcal{X}_{2,\mathrm{red}}$, or simply $\mathcal{X}$, be the reduced part of the Emerton-Gee stack for two-dimensional representations of G_K. It is defined over $\mathbb{F}$ and is an algebraic stack of pure dimension ef. The irreducible components of $\mathcal{X}$ are indexed by the non-Steinberg Serre weights. For a non-Steinberg Serre weight $V_{\vec{t},\vec{s}}$, we denote the corresponding irreducible component by $\mathcal{X}_{V_{\vec{t},\vec{s}}}$. If $\mathbb{F}'$ is a finite field extension of $\mathbb{F}$, then $\mathcal{X}_{V_{\vec{t},\vec{s}}}(\mathbb{F}')$ is the groupoid of representations $\overline{\rho} : \mathrm{G}_K \to \mathrm{GL}_2(\mathbb{F}')$ with $V_{\vec{t},\vec{s}} \in W(\overline{\rho})$.

We will consider the s_i's and t_i's associated to the Serre weight $V_{\vec{t},\vec{s}}$ to have indices in $\mathbb{Z}/f\mathbb{Z}$ via the identification of the set T with a set of representatives of $\mathbb{Z}/f\mathbb{Z}$. We will similarly consider the indexing set of the embeddings σ_i's to be $\mathbb{Z}/f\mathbb{Z}$.

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2. Extensions of Serre weights

Denote by Γ the group $\mathrm{GL}_2(k)$, by $\mathcal{K}$ the group $\mathrm{GL}_2(\mathcal{O}_K)$ and by $\mathcal{K}_n$ the group $1 + \pi^n M_2(\mathcal{O}_K)$ for $n \in \mathbb{Z}_{>0}$. Our objective in this section is to compute when extensions of Serre weights are non-trivial, for use later.

We will sometimes write $V_{\vec{t},\vec{s}}$ as $\bigotimes_{j=0}^{f-1}(\det^{t_j}\otimes \mathrm{Sym}^{s_j}\overline{\mathbb{F}}^2)^{Fr^{f-1-j}}$, where Γ acts on $\mathrm{Sym}^{s_j}\overline{\mathbb{F}}^2$ via the natural embedding $\Gamma\hookrightarrow \mathrm{GL}_2(\overline{\mathbb{F}})$ induced by σ_{f-1}. The exponentiation by Fr^{f-1-j} denotes precomposition of the action of Γ by the $(f-1-j)$-th power of the (arithmetic) Frobenius map.

Proposition 2.1. *The conditions for non-triviality of* $\mathrm{Ext}^1_\Gamma(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}})$ *are given as follows:*

(i) If $p>2$, $f>1$, then $\mathrm{Ext}^1_\Gamma(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}})\neq 0$ if and only if one of the following two conditions are satisfied:
  - *(a) $\exists j\in \mathbb{Z}/f\mathbb{Z}$ such that $s'_i=s_i$ for $i\neq j-1,j$; $s'_{j-1}=s_{j-1}-1$; $s'_j=p-s_j-2$; and $\sum_{i=0}^{f-1} t'_i p^{f-1-i}\equiv \sum_{i=0}^{f-1} t_i p^{f-1-i}+(s_j+1)p^{f-1-j} \mod p^f-1$.*
  - *(b) $\exists j\in \mathbb{Z}/f\mathbb{Z}$ such that $s'_i=s_i$ for $i\neq j-1,j$; $s'_{j-1}=s_{j-1}+1$; $s'_j=p-s_j-2$; and $\sum_{i=0}^{f-1} t'_i p^{f-1-i}\equiv \sum_{i=0}^{f-1} t_i p^{f-1-i}-(p-s_j-1)p^{f-1-j} \mod p^f-1$.*

(ii) If $p>2$, $f=1$, then $\mathrm{Ext}^1_\Gamma(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}})\neq 0$ if and only if one of the following two conditions are satisfied:
  - *(a) $s_0<p-2$; $s'_0=p-s_0-3$; and $t'_0\equiv t_0+s_0+1 \mod p-1$.*
  - *(b) $s_0\neq 0,p-1$; $s'_0=p-s_0-1$; and $t'_0\equiv t_0+s_0 \mod p-1$.*

(iii) If $p=2$, $f>1$, then $\mathrm{Ext}^1_\Gamma(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}})\neq 0$ if and only if the central characters for $V_{\vec{t},\vec{s}}$ and $V_{\vec{t'},\vec{s'}}$ are the same, as well as, $\exists j\in\mathbb{Z}/f\mathbb{Z}$ such that $s'_i=s_i$ for $i\neq j-1,j$; $s'_{j-1}=s_{j-1}\pm 1$; and $s'_j=p-s_j-2$.

(iv) If $p=2$, $f=1$, then $\mathrm{Ext}^1_\Gamma(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}})\neq 0$ if and only if $s'_0=s_0=0$; and $t'_0=t_0$.

Moreover, $\mathrm{Ext}^1_\Gamma(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}})$ *always has dimension* ≤ 1.

Proof. In order to compute $\mathrm{Ext}^1_\Gamma(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}})$, we need to compute the second socle layer of the injective hull of $V_{\vec{t'},\vec{s'}}$. Note that if an $\overline{\mathbb{F}}$-vector space V with Γ action is injective as an $\mathrm{SL}_2(k)$ module, then it is also injective as a Γ module. This is because any $\mathrm{SL}_2(k)$ module map ϕ can be lifted to a Γ module map by replacing it with $\frac{1}{[\Gamma:\mathrm{SL}_2(k)]}\sum_{g\in G/H} g(\phi)$. Therefore, we need to find a Γ module containing $V_{\vec{t'},\vec{s'}}$, so that it is the injective hull of $V_{\vec{t'},\vec{s'}}$ as an $\mathrm{SL}_2(k)$ module and compute its second socle layer.

Beyond this point, the steps are precisely as in [AJL], while carefully tracking through the twists by powers of the determinant. The final result (stated in the proposition) is then obtained in the same manner as [AJL, Cor. 4.5].

For $p=2$, note that if V is a non-trivial Γ extension of $V_{\vec{t},\vec{s}}$ by $V_{\vec{t'},\vec{s'}}$, then the central characters of $V_{\vec{t},\vec{s}}$ and $V_{\vec{t'},\vec{s'}}$ are the same (this holds true for any p). This is because $\overline{\mathbb{F}}[\mathcal{X}(\Gamma)]$ is semisimple, where $\mathcal{X}(\Gamma)$ is the center of Γ. Therefore, by twisting by a square root of the central character (possible since $p=2$), V can be assumed to be a non-trivial $\overline{\mathbb{F}}[\mathrm{PGL}_2(k)]=\overline{\mathbb{F}}[\mathrm{SL}_2(k)]$ extension. And therefore, $\mathrm{Ext}^1_\Gamma(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}})\neq 0$ implies that $\mathrm{Ext}^1_{\overline{\mathbb{F}}[\mathrm{SL}_2(k)]}(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}})\neq 0$. On the other hand, every non-trivial $\overline{\mathbb{F}}[\mathrm{SL}_2(k)]$ extension is a non-trivial $\overline{\mathbb{F}}[\mathrm{PGL}_2(k)]$ extension, and therefore, a non-trivial Γ extension. It follows that $\mathrm{Ext}^1_\Gamma(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}}) \neq 0 \iff \mathrm{Ext}^1_{\overline{\mathbb{F}}[\mathrm{SL}_2(k)]}(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}}) \neq 0$. The conditions for the latter are described in the paragraph preceding Corollary 4.5 in [AJL]. □

Remark 2.2. $\mathrm{Ext}^1_{\overline{\mathbb{F}}[\mathrm{SL}_2(k)]}(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}}) = 0$ implies $\mathrm{Ext}^1_\Gamma(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}}) = 0$. This is because any $\overline{\mathbb{F}}[\mathrm{SL}_2(k)]$ splitting ϕ can be upgraded to a Γ splitting by replacing it with $\frac{1}{[\Gamma:\mathrm{SL}_2(k)]}\sum_{g\in\Gamma/\mathrm{SL}_2(k)} g(\phi)$.

In order to compute $\mathrm{Ext}^1_{\mathcal{K}}(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}})$, we will use the Grothendieck spectral sequence. Let σ be a $\overline{\mathbb{F}}[\Gamma]$ representation, seen via inflation as a $\overline{\mathbb{F}}[\mathcal{K}]$ representation. The Grothendieck spectral sequence gives us the following left exact sequence:

$$0 \to \mathrm{Ext}^1_\Gamma(V_{\vec{t},\vec{s}},\sigma) \to \mathrm{Ext}^1_{\mathcal{K}}(V_{\vec{t},\vec{s}},\sigma) \to \mathrm{Hom}_\Gamma(V_{\vec{t},\vec{s}}, H^1(\mathcal{K}_1,\sigma)) \tag{2.2.1}$$

By [BP, Prop. 5.1], we have the following description of $H^1(\mathcal{K}_1,\sigma)$.

Proposition 2.3. *(i)*

$$H^1(\mathcal{K}_1,\sigma) \cong \bigoplus_{i=0}^{f-1} \sigma\otimes(V_2\otimes\det{}^{-1})^{Fr^{f-1-i}} \bigoplus_{i=1}^{d}\sigma$$

where V_2 is the subspace spanned by $\binom{2}{i}\tilde{x}^i\tilde{y}^{2-i}$ in $\mathrm{Sym}^2\overline{\mathbb{F}}^2$ where Γ acts via the embedding $\Gamma\hookrightarrow \mathrm{GL}_2(\overline{\mathbb{F}})$ induced by σ_{f-1};
$d = \dim_{\overline{\mathbb{F}}}\mathrm{Hom}(1+\pi\mathcal{O}_K,\overline{\mathbb{F}})$ for $p\neq 2$ and $d = \dim_{\overline{\mathbb{F}}}\mathrm{Hom}(1+\pi\mathcal{O}_K,\overline{\mathbb{F}}) - f$ for $p = 2$.

(ii) Under the above isomorphism, an element of $\sigma\otimes(V_2\otimes\det^{-1})^{Fr^{f-1-i}}$ can be seen explicitly as a map (cocycle) $\mathcal{K}_1\to\sigma$ via the following correspondence:

$$\alpha\otimes\tilde{x}^2 \in \sigma\otimes(V_2\otimes\det{}^{-1})^{Fr^{f-1-i}} \leftrightsquigarrow \kappa_i^l\alpha:\mathcal{K}_1\to\sigma$$
$$\alpha\otimes 2\tilde{x}\tilde{y} \in \sigma\otimes(V_2\otimes\det{}^{-1})^{Fr^{f-1-i}} \leftrightsquigarrow \epsilon_i\alpha:\mathcal{K}_1\to\sigma$$
$$\alpha\otimes\tilde{y}^2 \in \sigma\otimes(V_2\otimes\det{}^{-1})^{Fr^{f-1-i}} \leftrightsquigarrow \kappa_i^u\alpha:\mathcal{K}_1\to\sigma$$

where

$$\kappa_i^l : \begin{pmatrix}1+\pi a & \pi b\\ \pi c & 1+\pi d\end{pmatrix}\in\mathcal{K}_1 \mapsto \sigma_i(c)\in\overline{\mathbb{F}}$$
$$\epsilon_i : \begin{pmatrix}1+\pi a & \pi b\\ \pi c & 1+\pi d\end{pmatrix}\in\mathcal{K}_1 \mapsto \sigma_i(a-d)\in\overline{\mathbb{F}}$$
$$\kappa_i^u : \begin{pmatrix}1+\pi a & \pi b\\ \pi c & 1+\pi d\end{pmatrix}\in\mathcal{K}_1 \mapsto \sigma_i(a)\in\overline{\mathbb{F}}$$

(iii) The summand $\bigoplus_{i=1}^d\sigma\subset H^1(\mathcal{K}_1,\sigma)$ corresponds to maps $\mathcal{K}_1\to\sigma$ that factor through the determinant and are not given by any of the cocyles appearing in $\bigoplus_{i=0}^{f-1}\sigma\otimes(V_2\otimes\det^{-1})^{Fr^{f-1-i}}$.

Corollary 2.4. $\mathrm{Ext}^1_{\mathcal{K}}(V^\vee_{\vec{t},\vec{s}}, V^\vee_{\vec{t'},\vec{s'}})\neq 0$ if and only if $\mathrm{Ext}^1_{\mathcal{K}}(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}})\neq 0$.

Proof. Using (2.2.1), $\mathrm{Ext}^1_{\mathcal{K}}(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}})\neq 0$ implies that either $\mathrm{Ext}^1_\Gamma(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}})\neq 0$, or $\mathrm{Hom}_\Gamma(V_{\vec{t},\vec{s}}, H^1(\mathcal{K}_1, V_{\vec{t'},\vec{s'}})\neq 0$.

Either way, the central character is the same for $V_{\vec{t},\vec{s}}$ and $V_{\vec{t'},\vec{s'}}$. This is automatically true if $\mathrm{Ext}^1_\Gamma(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}}) \neq 0$ because the group algebra of the center of Γ is semisimple. If $\mathrm{Hom}_\Gamma(V_{\vec{t},\vec{s}}, H^1(\mathcal{K}_1, V_{\vec{t'},\vec{s'}})) \neq 0$, we use the description in Proposition 2.3 and the fact that $V_2 \otimes \det^{-1}$ has trivial central character. Therefore $\sum_{j=0}^{f-1} p^{f-1-j}(2t_j + s_j) \equiv \sum_{j=0}^{f-1} p^{f-1-j}(2t'_j + s'_j) \mod p^f - 1$.

Twisting by $\det^{-\sum_{j=0}^{f-1} p^{f-1-j}(2t_j+s_j)}$, we obtain:

$$\begin{aligned} &\mathrm{Ext}^1_{\mathcal{K}}(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}}) \neq 0 \\ \iff\ &\mathrm{Ext}^1_{\mathcal{K}}(V_{-\vec{t}-\vec{s},\vec{s}}, V_{-\vec{t'}-\vec{s'},\vec{s'}}) = \mathrm{Ext}^1_{\mathcal{K}}(V^\vee_{\vec{t},\vec{s}}, V^\vee_{\vec{t'},\vec{s'}}) \neq 0 \end{aligned}$$

□

Corollary 2.5. $\mathrm{Ext}^1_{\mathcal{K}}(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}}) \neq 0$ if and only if $\mathrm{Ext}^1_{\mathcal{K}}(V_{\vec{t'},\vec{s'}}, V_{\vec{t},\vec{s}}) \neq 0$.

Proof.

$$\begin{aligned} &\mathrm{Ext}^1_{\mathcal{K}}(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}}) \neq 0 \\ \iff\ &\mathrm{Ext}^1_{\mathcal{K}}(V^\vee_{\vec{t},\vec{s}}, V^\vee_{\vec{t'},\vec{s'}}) \neq 0 && \text{(by Corollary 2.4)} \\ \iff\ &\mathrm{Ext}^1_{\mathcal{K}}(V_{\vec{t'},\vec{s'}}, V_{\vec{t},\vec{s}}) \neq 0 && \text{(by taking duals)} \end{aligned}$$

□

Proposition 2.6. *Let $V_{\vec{t},\vec{s}}$ and $V_{\vec{t'},\vec{s'}}$ be a pair of non-isomorphic, non-Steinberg Serre weights. One shows up in the first $\mathcal{K}_1$ group cohomology of the other if and only if, after interchanging $V_{\vec{t},\vec{s}}$ and $V_{\vec{t'},\vec{s'}}$ if necessary, there exists $i \in \{0, ..., f-1\}$ such that*

(i) For $p > 2$,
- $s_i = s'_i + 2$,
- *For $j \neq i$, $s_j = s'_j$ and*
- $\sum_{j \in T} p^{f-1-j} t_j \equiv -p^{f-1-i} + \sum_{j\in T} p^{f-1-j} t'_j \mod p^f - 1$.

(ii) For $p = 2$,
- $s_i = s'_i + 1$,
- *For $j \neq i$, $s_j = s'_j$ and*
- *The central characters of the two Serre weights are the same.*

Further, for a pair of such non-isomorphic, non-Steinberg Serre weights, the multiplicity of appearance of one in the first $\mathcal{K}_1$ group cohomology of the other is at most 1.

Proof. The proof for $p > 2$ is covered by Proposition 5.4 and Corollary 5.5 in [BP]).

For $p = 2$, we first make the following observation. If $\mathrm{Hom}_\Gamma(V_{\vec{t},\vec{s}}, H^1(\mathcal{K}_1, V_{\vec{t'},\vec{s'}})) \neq 0$, we can twist both sides by the square root of the central character and obtain an inclusion of $V_{\vec{t},\vec{s}}$ into $H^1(\mathcal{K}_1, V_{\vec{t'},\vec{s'}})$ as $\mathrm{PGL}_2(k) = \mathrm{SL}_2(k)$ representations. On the other hand, suppose $V_{\vec{t},\vec{s}} \hookrightarrow H^1(\mathcal{K}_1, V_{\vec{t'},\vec{s'}})$ as $\mathrm{SL}_2(k)$ representations and the central characters of $V_{\vec{t},\vec{s}}$ and $V_{\vec{t'},\vec{s'}}$ are the same. Then this inclusion is easily seen to be an inclusion as Γ-representations.

Therefore, assuming the central characters of $V_{\vec{t},\vec{s}}$ and $V_{\vec{t'},\vec{s'}}$ are the same, we only need to find criteria for inclusion of $V_{\vec{t},\vec{s}}$ in $H^1(\mathcal{K}_1, V_{\vec{t'},\vec{s'}})$ as $\mathrm{SL}_2(k)$ representations.

To emphasize disregarding the determinant twists, we will denote by $L(\vec{r})$ or by $L(\sum p^{f-1-j} r_j)$ the irreducible $\mathrm{SL}_2(k)$ representation $\otimes_{j=0}^{f-1}(\mathrm{Sym}^{r_j}\overline{\mathbb{F}}^2)^{Fr^{f-1-j}}$ where $r_j \in [0, p-1]$ for each j and $\mathrm{SL}_2(k)$ acts on $\mathrm{Sym}^{r_j}\overline{\mathbb{F}}^2$ via $\sigma_{f-1} : \mathrm{SL}_2(k) \hookrightarrow \mathrm{SL}_2(\overline{\mathbb{F}})$.

By Proposition 2.3, $H^1(\mathcal{K}_1, \sigma) \cong \bigoplus_{i=0}^{f-1}(L(\vec{s'}) \otimes V_2^{Fr^{f-2-i}}) \bigoplus_{i=1}^{d} L(\vec{s'})$ as $\mathrm{SL}_2(k)$ representations. As $L(\vec{s}) \ncong L(\vec{s'})$, we need to understand when $L(\vec{s})$ embeds into $L(\vec{s'}) \otimes V_2^{Fr^{f-2-i}} \cong L(\vec{s'}) \otimes L(1)^{Fr^{f-1-i}}$ for a given i.

(1) $s'_i = 0$. Then $L(\vec{s'}) \otimes L(1)^{Fr^{f-1-i}}$ is irreducible and isomorphic to $L(\vec{s})$, where $s_i = 1$ and $s_j = s'_j$ for $j \neq i$.
(2) $s'_i = 1$. Then

$$\begin{aligned} L(\vec{s'}) \otimes L(1)^{Fr^{f-1-i}} &\cong L(s'_0)^{Fr^{f-1-0}} \otimes \cdots \otimes L(s'_{i-1})^{Fr^{f-i}} \otimes (L(1)\otimes L(1))^{Fr^{f-1-i}} \\ &\qquad \otimes L(s'_{i+1})^{Fr^{f-2-i}} \otimes \cdots \otimes L(s'_{f-1})^{Fr^0} \\ &\cong L(s'_0)^{Fr^{f-1-0}} \otimes \cdots \otimes L(s'_{i-1})^{Fr^{f-i}} \otimes Q_1(0)^{Fr^{f-1-i}} \\ &\qquad \otimes L(s'_{i+1})^{Fr^{f-2-i}} \otimes \cdots \otimes L(s'_{f-1})^{Fr^0} \end{aligned}$$

where $Q_1(0)$ is a self-dual representation of Loewy length 3, with composition factors $L(0)$, $L(2)$, $L(0)$ by [AJL, Lem. 3.1]. In fact, [AJL, Lem. 3.1] says that $Q_1(0)$ is a direct summand of $(\mathrm{Sym}^1\overline{\mathbb{F}}^2 \otimes \mathrm{Sym}^1\overline{\mathbb{F}}^2)^{Fr^{f-1-i}}$, but by comparing dimensions, they are equal. As $\mathrm{SL}_2(\overline{\mathbb{F}})$ representations:

$$\begin{aligned} & L(s'_0)^{Fr^{f-1-0}} \otimes \cdots \otimes L(s'_{i-1})^{Fr^{f-i}} \otimes L(0)^{Fr^{f-1-i}} \\ &\qquad \otimes L(s'_{i+1})^{Fr^{f-2-i}} \otimes \cdots \otimes L(s'_{f-1})^{Fr^0} \\ \hookrightarrow\ & \mathrm{soc}\Big(L(s'_0)^{Fr^{f-1-0}} \otimes \cdots \otimes L(s'_{i-1})^{Fr^{f-i}} \otimes Q_1(0)^{Fr^{f-1-i}} \\ &\qquad \otimes L(s'_{i+1})^{Fr^{f-2-i}} \otimes \cdots \otimes L(s'_{f-1})^{Fr^0}\Big) \\ \hookrightarrow\ & \mathrm{soc}\Big(Q_1(s'_0)^{Fr^{f-1-0}} \otimes \cdots \otimes Q_1(s'_{i-1})^{Fr^{f-i}} \otimes Q_1(0)^{Fr^{f-1-i}} \\ &\qquad \otimes Q_1(s'_{i+1})^{Fr^{f-2-i}} \otimes \cdots \otimes Q_1(s'_{f-1})^{Fr^0}\Big) \\ \cong\ & L(s'_0)^{Fr^{f-1-0}} \otimes \cdots \otimes L(s'_{i-1})^{Fr^{f-i}} \otimes L(0)^{Fr^{f-1-i}} \\ &\qquad \otimes L(s'_{i+1})^{Fr^{f-2-i}} \otimes \cdots \otimes L(s'_{f-1})^{Fr^0} \end{aligned}$$

The isomorphism in the last step is by [AJL, Lem. 3.4]. Using [AJL, Cor. 4.2] and the assumption that $L(\vec{s})$ is not Steinberg, we conclude that $L(\vec{s})$ embeds into $L(\vec{s'}) \otimes L(1)^{Fr^{f-1-i}}$ if and only if $L(\vec{s}) \cong L(s'_0)^{Fr^{f-1-0}} \otimes \cdots \otimes L(s'_{i-1})^{Fr^{f-i}} \otimes L(0)^{Fr^{f-1-i}} \otimes L(s'_{i+1})^{Fr^{f-2-i}} \otimes \cdots \otimes L(s'_{f-1})^{Fr^0}$.

□

Remark 2.7. The explicit calculation of $H^1(\mathcal{K}_1, V_{\vec{t'},\vec{s'}})$ in [BP, Prop. 5.1] shows that the inflation homomorphism $H^1(\mathcal{K}_1/\mathcal{K}_2, V_{\vec{t'},\vec{s'}}) \to H^1(\mathcal{K}_1, V_{\vec{t'},\vec{s'}})$ is an isomorphism, since each $\mathcal{K}_1$ cocycle is in fact a $\mathcal{K}_1/\mathcal{K}_2$ cocycle. By the construction of the

Grothendieck spectral sequence, this means exactly that

$$\mathrm{soc}_\Gamma\left(\frac{\mathrm{inj}_{\mathcal{K}}(V_{\vec{t'},\vec{s'}})^{\mathcal{K}_2}/V_{\vec{t'},\vec{s'}}}{\mathrm{inj}_{\mathcal{K}}(V_{\vec{t'},\vec{s'}})^{\mathcal{K}_1}/V_{\vec{t'},\vec{s'}}}\right) = \mathrm{soc}_\Gamma\left(\frac{\mathrm{inj}_{\mathcal{K}}(V_{\vec{t'},\vec{s'}})/V_{\vec{t'},\vec{s'}}}{\mathrm{inj}_{\mathcal{K}}(V_{\vec{t'},\vec{s'}})^{\mathcal{K}_1}/V_{\vec{t'},\vec{s'}}}\right)$$

Lemma 2.8. *Let $V_{\vec{t},\vec{s}}$ and $V_{\vec{t'},\vec{s'}}$ be a pair of Serre weights. Then the natural map $\mathrm{Ext}^1_{\mathcal{K}/\mathcal{K}_2}(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}}) \to \mathrm{Ext}^1_{\mathcal{K}}(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}})$ is an isomorphism.*

Proof. For a group G with an $\overline{\mathbb{F}}$-representation σ, let $\mathrm{inj}_G(\sigma)$ denote the injective hull of σ as a smooth $\overline{\mathbb{F}}[G]$ module. Then, for each Serre weight σ, $\mathrm{inj}_{\mathcal{K}}(\sigma)^{\mathcal{K}_n}$ is an injective $\mathcal{K}/\mathcal{K}_n$ module. By injectivity of $\mathrm{inj}_{\mathcal{K}/\mathcal{K}_n}(\sigma)$, there exists a map $\mathrm{inj}_{\mathcal{K}}(\sigma)^{\mathcal{K}_n} \to \mathrm{inj}_{\mathcal{K}/\mathcal{K}_n}(\sigma)$. The kernel of this map must be trivial, by the hull property of $\mathrm{inj}_{\mathcal{K}}(\sigma)$. By the hull property of $\mathrm{inj}_{\mathcal{K}/\mathcal{K}_n}(\sigma)$, it is forced to be an isomorphism. We will henceforth use $\mathrm{inj}_{\mathcal{K}}(\sigma)^{\mathcal{K}_n}$ as the injective hull of σ as a $\mathcal{K}/\mathcal{K}_n$ representation.

The explicit calculation of $H^1(\mathcal{K}_1, V_{\vec{t'},\vec{s'}})$ in [BP, Prop. 5.1] shows that the inflation homomorphism $H^1(\mathcal{K}_1/\mathcal{K}_2, V_{\vec{t'},\vec{s'}}) \to H^1(\mathcal{K}_1, V_{\vec{t'},\vec{s'}})$ is an isomorphism, since each $\mathcal{K}_1$ cocycle is in fact a $\mathcal{K}_1/\mathcal{K}_2$ cocycle. By the construction of the Grothendieck spectral sequence, this means exactly that

$$\mathrm{soc}_\Gamma\left(\frac{\left(\mathrm{inj}_{\mathcal{K}}(V_{\vec{t'},\vec{s'}})^{\mathcal{K}_2}/V_{\vec{t'},\vec{s'}}\right)^{\mathcal{K}_1}}{\mathrm{inj}_{\mathcal{K}}(V_{\vec{t'},\vec{s'}})^{\mathcal{K}_1}/V_{\vec{t'},\vec{s'}}}\right) = \mathrm{soc}_\Gamma\left(\frac{\left(\mathrm{inj}_{\mathcal{K}}(V_{\vec{t'},\vec{s'}})/V_{\vec{t'},\vec{s'}}\right)^{\mathcal{K}_1}}{\mathrm{inj}_{\mathcal{K}}(V_{\vec{t'},\vec{s'}})^{\mathcal{K}_1}/V_{\vec{t'},\vec{s'}}}\right) \tag{2.8.1}$$

Now, suppose $V_{\vec{t},\vec{s}}$ lives inside the Γ socle of $\frac{\left(\mathrm{inj}_{\mathcal{K}}(V_{\vec{t'},\vec{s'}})^{\mathcal{K}_2}/V_{\vec{t'},\vec{s'}}\right)^{\mathcal{K}_1}}{\mathrm{inj}_{\mathcal{K}}(V_{\vec{t'},\vec{s'}})^{\mathcal{K}_1}/V_{\vec{t'},\vec{s'}}}$ with multiplicity n. Equivalently, $\mathrm{Hom}_\Gamma(V_{\vec{t},\vec{s}}, H^1(\mathcal{K}/\mathcal{K}_2, V_{\vec{t'},\vec{s'}}))$ is n-dimensional. Let N denote the preimage in $\mathrm{inj}_{\mathcal{K}}(V_{\vec{t'},\vec{s'}})^{\mathcal{K}_2}/V_{\vec{t'},\vec{s'}}$ of $V_{\vec{t},\vec{s}}^{\oplus n} \subset \mathrm{soc}_\Gamma\left(\frac{\left(\mathrm{inj}_{\mathcal{K}}(V_{\vec{t'},\vec{s'}})^{\mathcal{K}_2}/V_{\vec{t'},\vec{s'}}\right)^{\mathcal{K}_1}}{\mathrm{inj}_{\mathcal{K}}(V_{\vec{t'},\vec{s'}})^{\mathcal{K}_1}/V_{\vec{t'},\vec{s'}}}\right)$.

Suppose, further, that $L \cong V_{\vec{t},\vec{s}}^{\oplus l} \subset \mathrm{soc}_\Gamma\left(\mathrm{inj}_{\mathcal{K}}(V_{\vec{t'},\vec{s'}})/V_{\vec{t'},\vec{s'}}\right) - \mathrm{soc}_\Gamma\left(\mathrm{inj}_{\mathcal{K}}(V_{\vec{t'},\vec{s'}})^{\mathcal{K}_2}/V_{\vec{t'},\vec{s'}}\right)$. Then, the preimage inside $\mathrm{inj}_{\mathcal{K}}(V_{\vec{t'},\vec{s'}})/V_{\vec{t'},\vec{s'}}$ of $V_{\vec{t},\vec{s}}^{\oplus n} \subset \mathrm{soc}_\Gamma\left(\frac{\left(\mathrm{inj}_{\mathcal{K}}(V_{\vec{t'},\vec{s'}})/V_{\vec{t'},\vec{s'}}\right)^{\mathcal{K}_1}}{\mathrm{inj}_{\mathcal{K}}(V_{\vec{t'},\vec{s'}})^{\mathcal{K}_1}/V_{\vec{t'},\vec{s'}}}\right)$ contains $L+N$. As $L \not\subset N$, $L+N = L \oplus N$, and the multiplicity of $V_{\vec{t},\vec{s}}$ in $\mathrm{soc}_\Gamma\left(\frac{\left(\mathrm{inj}_{\mathcal{K}}(V_{\vec{t'},\vec{s'}})/V_{\vec{t'},\vec{s'}}\right)^{\mathcal{K}_1}}{\mathrm{inj}_{\mathcal{K}}(V_{\vec{t'},\vec{s'}})^{\mathcal{K}_1}/V_{\vec{t'},\vec{s'}}}\right)$ is $\geq l+n$, implying $l=0$ by (2.8.1). Therefore, $\mathrm{soc}_\Gamma(\mathrm{inj}_{\mathcal{K}}(V_{\vec{t'},\vec{s'}})^{\mathcal{K}_2}/V_{\vec{t'},\vec{s'}}) \hookrightarrow \mathrm{soc}_\Gamma(\mathrm{inj}_{\mathcal{K}}(V_{\vec{t'},\vec{s'}})/V_{\vec{t'},\vec{s'}})$ is an equality.

□

Since $\mathcal{K}/\mathcal{K}_2$ is a finite group, $\mathrm{Ext}^1_{\mathcal{K}/\mathcal{K}_2}(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}}) \cong H^1(\mathcal{K}/\mathcal{K}_2, V_{\vec{t},\vec{s}}^\vee \otimes V_{\vec{t'},\vec{s'}})$. The Grothendieck spectral sequence gives us the following left exact sequence:

(2.8.2)

$$0 \to H^1(\Gamma, V_{\vec{t},\vec{s}}^\vee \otimes V_{\vec{t'},\vec{s'}}) \xrightarrow{\mathrm{inf}} H^1(\mathcal{K}/\mathcal{K}_2, V_{\vec{t},\vec{s}}^\vee \otimes V_{\vec{t'},\vec{s'}}) \xrightarrow{\mathrm{res}} H^1(\mathcal{K}_1/\mathcal{K}_2, V_{\vec{t},\vec{s}}^\vee \otimes V_{\vec{t'},\vec{s'}})^\Gamma$$

Proposition 2.9. *Suppose $e > 1$. Then the* res *map in (2.8.2) is a split surjection.*

Proof. $e > 1$ implies that $p \in \pi^2\mathcal{O}_K$. Let $\mathcal{O}_K^{ur}$ be the ring of integers for the maximal unramified subextension inside E over $\mathbb{Q}_p$. Therefore $k \cong \mathcal{O}_K^{ur}/p \hookrightarrow \mathcal{O}_K/\pi^2$. This gives a splitting of the natural surjection $\mathrm{GL}_2(\mathcal{O}_K/\pi^2) \twoheadrightarrow \mathrm{GL}_2(k) \cong \mathrm{GL}_2(\mathcal{O}_K^{ur}/p)$. We obtain the following split exact sequence:

$$1 \longrightarrow \mathcal{K}_1/\mathcal{K}_2 \longrightarrow \mathcal{K}/\mathcal{K}_2 \longrightarrow \Gamma \longrightarrow 1$$

Therefore, $\mathcal{K}/\mathcal{K}_2 \cong \mathcal{K}_1/\mathcal{K}_2 \rtimes \Gamma$. For $b \in \Gamma$ and $a \in \mathcal{K}_1/\mathcal{K}_2$, denote bab^{-1} by a^b.

Suppose σ is a Γ representation (seen via inflation as a $\mathcal{K}/\mathcal{K}_2$ representation) and ψ is a cocycle representing a nonzero element of $H^1(\mathcal{K}_1/\mathcal{K}_2, \sigma)^\Gamma$. As $\mathcal{K}_1/\mathcal{K}_2$ action is trivial on σ, $H^1(\mathcal{K}_1/\mathcal{K}_2, \sigma)^\Gamma = Z^1(\mathcal{K}_1/\mathcal{K}_2, \sigma)^\Gamma$. Γ-invariance means precisely that for $b \in \Gamma$ and $a \in \mathcal{K}_1/\mathcal{K}_2$, $b^{-1}\psi(a^b) = \psi(a)$.

We define a function δ on $\mathcal{K}_1/\mathcal{K}_2 \rtimes \Gamma$ by setting $\delta((a,b))$ equal to $\psi(a)$. I claim that δ is a cocyle, i.e., $\delta((a,b)(a',b')) = \delta((a,b)) + (a,b)\cdot\delta((a',b'))$. Evaluation of the left hand side gives us:

$$\begin{aligned} L.H.S. &= \delta((aa'^b, bb')) \\ &= \psi(aa'^b) \\ &= \psi(a) + \psi(a'^b) \end{aligned}$$

Evaluation of the right hand side gives us:

$$\begin{aligned} R.H.S. &= \psi(a) + (a,1)(1,b)\cdot\psi(a') \\ &= \psi(a) + (1,b)\cdot\psi(a') \\ &= \psi(a) + (1,b)\cdot((1,b^{-1})\cdot\psi(a'^b)) && \text{(as } \psi \text{ is } \Gamma\text{-invariant)} \\ &= \psi(a) + \psi(a'^b) \\ &= L.H.S. \end{aligned}$$

This establishes that δ is a cocyle and therefore, res map in (2.8.2) is a split surjection.

□

Corollary 2.10. If $e > 1$,

$$\mathrm{Ext}^1_{\mathcal{K}}(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}}) = \mathrm{Ext}^1_{\Gamma}(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}}) \oplus \mathrm{Hom}_\Gamma(V_{\vec{t},\vec{s}}, H^1(\mathcal{K}_1/\mathcal{K}_2, V_{\vec{t'},\vec{s'}})).$$

Proof. This is immediate from Proposition 2.9 and the fact that $H^1(\mathcal{K}_1/\mathcal{K}_2, V_{\vec{t},\vec{s}}^\vee \otimes V_{\vec{t'},\vec{s'}})^\Gamma \cong \mathrm{Hom}_\Gamma(V_{\vec{t},\vec{s}}, H^1(\mathcal{K}_1/\mathcal{K}_2, V_{\vec{t'},\vec{s'}}))$ by the explicit description in Proposition 2.3. □

Lemma 2.11. *Let $p > 2$, $r \le p-3$. Then the following are true:*

(i) $\mathrm{Sym}^{r+2}\overline{\mathbb{F}}^2$ *embeds into* $\mathrm{Sym}^2\overline{\mathbb{F}}^2 \otimes \mathrm{Sym}^r\overline{\mathbb{F}}^2$ *as a direct summand of multiplicity* 1.

(ii) Let the obvious basis of $\mathrm{Sym}^{r+2}\overline{\mathbb{F}}^2$ *be given by* $\{w^k z^{r+2-k}\}_{k\in[0,r+2]}$. *Further, let a basis of* $\mathrm{Sym}^2\overline{\mathbb{F}}^2 \otimes \mathrm{Sym}^r\overline{\mathbb{F}}^2$ *be given by* $\{\tilde{x}^j\tilde{y}^{2-j}\otimes x^k y^{r-k}\}_{(j,k)\in[0,2]\times[0,r]}$ *if* $r > 0$, *and by* $\{\tilde{x}^j\tilde{y}^{2-j}\otimes 1\}_{j\in[0,2]}$ *if* $r = 0$. *The embedding is given*

(uniquely upto scalar multiplication) as follows:

$$\begin{array}{rcrr} w^k z^{r+2-k} & \mapsto & \tilde{x}^2 \otimes \frac{k(k-1)}{(r+2)(r+1)} x^{k-2} y^{r+2-k} & \\ & & +2\tilde{x}\tilde{y} \otimes \frac{k(r+2-k)}{(r+2)(r+1)} x^{k-1} y^{r+1-k} & \\ & & +\tilde{y}^2 \otimes \frac{(r+2-k)(r+1-k)}{(r+2)(r+1)} x^k y^{r-k} & \text{for } k \in [2,r] \\ wz^{r+1} & \mapsto & 2\tilde{x}\tilde{y} \otimes \frac{1}{r+2} y^r + \tilde{y}^2 \otimes \frac{r}{r+2} x y^{r-1} & \text{if } r > 0 \\ w^{r+1} z & \mapsto & \tilde{x}^2 \otimes \frac{r}{r+2} x^{r-1} y + 2\tilde{x}\tilde{y} \otimes \frac{1}{r+2} x^r & \text{if } r > 0 \\ wz & \mapsto & \tilde{x}\tilde{y} \otimes 1 & \text{if } r = 0 \\ w^{r+2} & \mapsto & \tilde{x}^2 \otimes x^r & \\ z^{r+2} & \mapsto & \tilde{y}^2 \otimes y^r & \end{array}$$

Proof. The first statement is from [BP, Prop. 5.4]. The second statement can be verified by direct computation. □

Lemma 2.12. *Let $V_{\vec{t},\vec{s}}$, a Serre weight. Denote by $\{\otimes_{j=0}^{f-1} w^{k_j} z^{s_j-k_j}\}_{(k_j)_j}$ the obvious basis of $V_{\vec{t},\vec{s}}$. Use the same notation to denote a basis of $V_{-\vec{t}-s,\vec{s}}$. Then $V_{\vec{t},\vec{s}}^\vee \cong V_{-\vec{t}-s,\vec{s}}$ under the following map:*

$$\begin{array}{ccc} V_{\vec{t},\vec{s}}^\vee & \to & V_{-\vec{t}-s,\vec{s}} \\ \otimes_j (w^{k_j} z^{s_j-k_j})^\vee & \mapsto & \otimes_j \binom{s_j}{k_j} w^{s_j-k_j} (-z)^{k_j} \end{array}$$

Proof. By direct computation. □

Lemma 2.13. *Let $p > 2$. Consider a pair of non-isomorphic, non-Steinberg Serre weights $V_{\vec{t},\vec{s}}$ and $V_{\vec{t}',\vec{s}'}$ satisfying the condition in Proposition 2.6, that is, $s_i = s_i' + 2$, $s_j = s_j'$ for $j \neq i$ and $\sum_{j \in T} p^{f-1-j} t_j \equiv -p^{f-1-i} + \sum_{j \in T} p^{f-1-j} t_j' \mod p^f - 1$.*

Denote by $\otimes_{j=0}^{f-1} (w^{k_j} z^{s_j-k_j})^\vee$ the dual of $\otimes_{j=0}^{f-1} w^{k_j} z^{s_j-k_j}$, where $\{\otimes_{j=0}^{f-1} w^{k_j} z^{s_j-k_j}\}_{(k_j)_j}$ gives a basis of $V_{\vec{t},\vec{s}}$. Let the basis of $V_{\vec{t}',\vec{s}'}$ be given by $\{\otimes_{j=0}^{f-1} x^{k_j'} y^{s_j'-k_j'}\}_{(k_j')_j}$.

Then the Γ-invariant cocyles of $H^1(\mathcal{K}_1/\mathcal{K}_2, V_{\vec{t},\vec{s}}^\vee \otimes V_{\vec{t}',\vec{s}'}) \cong H^1(\mathcal{K}_1/\mathcal{K}_2, V_{-\vec{t}-\vec{s},\vec{s}} \otimes V_{\vec{t}',\vec{s}'})$ are a 1-dimensional subspace spanned by

$$\kappa_i^l A + \epsilon_i B + \kappa_i^u C$$

where κ_i^l, ϵ_i and κ_i^u are homomorphisms $\mathcal{K}_1/\mathcal{K}_2 \to \overline{\mathbb{F}}$ defined in Proposition 2.3 and A, B and C are elements of $V_{\vec{t},\vec{s}}^\vee \otimes V_{\vec{t}',\vec{s}'}$ defined below.

For $s_i' > 0$,

$$
\begin{aligned}
A &= -\left(\sum_{(k_j)_{j\neq i}} (\otimes_{j\neq i}\binom{s_j}{k_j} w^{s_j-k_j}(-z)^{k_j}) \otimes (\otimes_{j\neq i} x^{k_j} y^{s'_j-k_j})\right) \bigotimes \\
&\left(\sum_{k_i=2}^{s'_i}\left(\binom{s_i}{k_i} w^{s_i-k_i}(-z)^{k_i} \otimes \frac{k_i(k_i-1)}{(s'_i+2)(s'_i+1)} x^{k_i-2} y^{s'_i+2-k_i}\right)\right. \\
&\qquad + \left(s_i w(-z)^{s_i-1} \otimes \frac{s'_i}{s'_i+2} x^{s'_i-1} y\right) \\
&\qquad \left. + \left((-z)^{s_i} \otimes x^{s'_i}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
B &= \left(\sum_{(k_j)_{j\neq i}} (\otimes_{j\neq i}\binom{s_j}{k_j} w^{s_j-k_j}(-z)^{k_j}) \otimes (\otimes_{j\neq i} x^{k_j} y^{s'_j-k_j})\right) \bigotimes \\
&\left(\sum_{k_i=2}^{s'_i}\left(\binom{s_i}{k_i} w^{s_i-k_i}(-z)^{k_i} \otimes \frac{k_i(s'_i+2-k_i)}{(s'_i+2)(s'_i+1)} x^{k_i-1} y^{s'_i+1-k_i}\right)\right. \\
&\qquad + \left(s_i w^{s_i-1}(-z) \otimes \frac{1}{s'_i+2} y^{s'_i}\right) \\
&\qquad \left. + \left(s_i w(-z)^{s_i-1} \otimes \frac{1}{s'_i+2} x^{s'_i}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
C &= \left(\sum_{(k_j)_{j\neq i}} (\otimes_{j\neq i}\binom{s_j}{k_j} w^{s_j-k_j}(-z)^{k_j}) \otimes (\otimes_{j\neq i} x^{k_j} y^{s'_j-k_j})\right) \bigotimes \\
&\left(\sum_{k_i=2}^{s'_i}\left(\binom{s_i}{k_i} w^{s_i-k_i}(-z)^{k_i} \otimes \frac{(s'_i+2-k_i)(s'_i+1-k_i)}{(s'_i+2)(s'_i+1)} x^{k_i} y^{s'_i-k_i}\right)\right. \\
&\qquad + \left(s_i w^{s_i-1}(-z) \otimes \frac{s'_i}{s'_i+2} x y^{s'_i-1}\right) \\
&\qquad \left. + \left((w^{s_i} \otimes y^{s'_i}\right)\right)
\end{aligned}
$$

For $s'_i = 0$,

$$
\begin{aligned}
A &= -\left(\sum_{(k_j)_{j\neq i}} (\otimes_{j\neq i}\binom{s_j}{k_j} w^{s_j-k_j}(-z)^{k_j}) \otimes (\otimes_{j\neq i} x^{k_j} y^{s'_j-k_j})\right) \bigotimes ((-z)^{s_i} \otimes 1) \\
B &= \left(\sum_{(k_j)_{j\neq i}} (\otimes_{j\neq i}\binom{s_j}{k_j} w^{s_j-k_j}(-z)^{k_j}) \otimes (\otimes_{j\neq i} x^{k_j} y^{s'_j-k_j})\right) \bigotimes (-wz \otimes 1) \\
C &= \left(\sum_{(k_j)_{j\neq i}} (\otimes_{j\neq i}\binom{s_j}{k_j} w^{s_j-k_j}(-z)^{k_j}) \otimes (\otimes_{j\neq i} x^{k_j} y^{s'_j-k_j})\right) \bigotimes (w^{s_i} \otimes 1)
\end{aligned}
$$

Proof. The first step is to compute $\left(H^1(\mathcal{K}_1/\mathcal{K}_2, V_{\vec{t},\vec{s}}^\vee \otimes V_{\vec{t'},\vec{s'}})\right)^\Gamma$. By Proposition 2.3, this group is isomorphic to $\mathrm{Hom}_\Gamma(V_{\vec{t},\vec{s}}, \bigoplus_{i=0}^{f-1} V_{\vec{t'},\vec{s'}} \otimes (V_2 \otimes \det^{-1})^{Fr^{f-1-i}} \bigoplus_{i=1}^{d} V_{\vec{t'},\vec{s'}})$. Using Lemma 2.11, $V_{\vec{t},\vec{s}}$ has a unique (upto scalars) embedding into $\bigoplus_{i=0}^{f-1} V_{\vec{t'},\vec{s'}} \otimes (V_2 \otimes \det^{-1})^{Fr^{f-1-i}} \bigoplus_{i=1}^{d} V_{\vec{t'},\vec{s'}}$. This embedding may be written as an element of $V_{\vec{t},\vec{s}}^\vee \otimes V_{\vec{t'},\vec{s'}} \otimes (V_2 \otimes \det^{-1})^{Fr^{f-1-i}} \subset H^1(\mathcal{K}_1/\mathcal{K}_2, V_{\vec{t},\vec{s}}^\vee \otimes V_{\vec{t'},\vec{s'}})$. Employing Proposition 2.3 to further write this element as an explicit map $\mathcal{K}_1/\mathcal{K}_2 \to V_{\vec{t},\vec{s}}^\vee \otimes V_{\vec{t'},\vec{s'}}$, we obtain the following values of A, B and C:

For $s'_i > 0$,

$$A = -\left(\sum_{(k_j)_{j\neq i}} (\otimes_{j\neq i}(w^{k_j} z^{s_j-k_j})^\vee) \otimes (\otimes_{j\neq i} x^{k_j} y^{s'_j-k_j})\right) \bigotimes \left(\sum_{k_i=2}^{s'_i}\left((w^{k_i} z^{s_i-k_i})^\vee \otimes \frac{k_i(k_i-1)}{(s'_i+2)(s'_i+1)} x^{k_i-2} y^{s'_i+2-k_i}\right) + \left((w^{s_i-1}z)^\vee \otimes \frac{s'_i}{s'_i+2} x^{s'_i-1} y\right) + \left((w^{s_i})^\vee \otimes x^{s'_i}\right)\right)$$

$$B = \left(\sum_{(k_j)_{j\neq i}} (\otimes_{j\neq i}(w^{k_j} z^{s_j-k_j})^\vee) \otimes (\otimes_{j\neq i} x^{k_j} y^{s'_j-k_j})\right) \bigotimes \left(\sum_{k_i=2}^{s'_i}\left((w^{k_i} z^{s_i-k_i})^\vee \otimes \frac{k_i(s'_i+2-k_i)}{(s'_i+2)(s'_i+1)} x^{k_i-1} y^{s'_i+1-k_i}\right) + \left((wz^{s_i-1})^\vee \otimes \frac{1}{s'_i+2} y^{s'_i}\right) + \left((w^{s_i-1}z)^\vee \otimes \frac{1}{s'_i+2} x^{s'_i}\right)\right)$$

$$C = \left(\sum_{(k_j)_{j\neq i}} (\otimes_{j\neq i}(w^{k_j} z^{s_j-k_j})^\vee) \otimes (\otimes_{j\neq i} x^{k_j} y^{s'_j-k_j})\right) \bigotimes \left(\sum_{k_i=2}^{s'_i}\left((w^{k_i} z^{s_i-k_i})^\vee \otimes \frac{(s'_i+2-k_i)(s'_i+1-k_i)}{(s'_i+2)(s'_i+1)} x^{k_i} y^{s'_i-k_i}\right) + \left(wz^{s_i-1})^\vee \otimes \frac{s'_i}{s'_i+2} xy^{s'_i-1}\right) + \left((z^{s_i})^\vee \otimes y^{s'_i}\right)\right)$$

For $s'_i = 0$,

$$\begin{aligned} A &= -\left(\sum_{(k_j)_{j\neq i}} (\otimes_{j\neq i}(w^{k_j}z^{s_j-k_j})^\vee)\otimes(\otimes_{j\neq i}x^{k_j}y^{s'_j-k_j})\right)\bigotimes((w^{s_i})^\vee\otimes 1)\\ B &= \left(\sum_{(k_j)_{j\neq i}} (\otimes_{j\neq i}(w^{k_j}z^{s_j-k_j})^\vee)\otimes(\otimes_{j\neq i}x^{k_j}y^{s'_j-k_j})\right)\bigotimes\left((wz)^\vee\otimes\frac{1}{2}\right)\\ C &= \left(\sum_{(k_j)_{j\neq i}} (\otimes_{j\neq i}(w^{k_j}z^{s_j-k_j})^\vee)\otimes(\otimes_{j\neq i}x^{k_j}y^{s'_j-k_j})\right)\bigotimes((z^{s_i})^\vee\otimes 1) \end{aligned}$$

Using Lemma 2.12, we can rewrite elements in $V_{\vec{t},\vec{s}}^\vee$ as elements of $V_{-\vec{t}-\vec{s},\vec{s}}$, giving us the desired answer. □

Our next order of business is to check if a Γ-invariant cocycle in $H^1(\mathcal{K}_1/\mathcal{K}_2, V_{\vec{t},\vec{s}}^\vee\otimes V_{\vec{t'},\vec{s'}})\cong H^1(\mathcal{K}_1/\mathcal{K}_2, V_{-\vec{t}-\vec{s},\vec{s}}\otimes V_{\vec{t'},\vec{s'}})$ is in the image of the res map in (2.8.2). Therefore, we will try and extend such a cocycle to $\mathcal{K}$. However, instead of extending it to all of $\mathcal{K}$, we will first focus our attention on extending it to the subgroup of $\mathcal{K}$ generated by the upper unipotent and diagonal matrices.

Proposition 2.14. *Let $e=1$. Consider a pair of non-isomorphic, weakly regular Serre weights $V_{\vec{t},\vec{s}}$ and $V_{\vec{t'},\vec{s'}}$ satisfying the conditions in Proposition 2.6. In particular,*

$$\mathrm{Hom}_\Gamma(V_{\vec{t},\vec{s}}, H^1(\mathcal{K}_1, V_{\vec{t'},\vec{s'}})\neq 0$$

Then res *is the zero map and* inf *in (2.8.2) is an isomorphism, implying that $\mathrm{Ext}^1_{\mathcal{K}}(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}})\neq 0$ iff $\mathrm{Ext}^1_\Gamma(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}})\neq 0$.*

Proof. We will show the following stronger result. Let $p>2$, and let $V_{\vec{t},\vec{s}}$ and $V_{\vec{t'},\vec{s'}}$ satisfy:

$$\begin{aligned} &s_i = s'_i+2,\\ &s_j = s'_j \text{ for } j\neq i,\\ &s'_{i+1}<p-1, \text{ and}\\ &\sum_{j\in T} p^{f-1-j}t_j \equiv -p^{f-1-i}+\sum_{j\in T}p^{f-1-j}t'_j \mod p^f-1. \end{aligned}$$

Then res is the zero map and inf in (2.8.2) is an isomorphism. (Note that the above conditions on p, $\vec{s}$ and $\vec{s'}$ are automatically implied by the hypotheses in the statement of the Proposition.)

We will assume without loss of generality that $\vec{t'}=\vec{0}$.

Our proof will show that there is no way to extend a Γ-invariant cocycle $\psi\in H^1(\mathcal{K}_1/\mathcal{K}_2, V_{-\vec{t}-\vec{s},\vec{s}}\otimes V_{\vec{t'},\vec{s'}})$ simultaneously to upper unipotent and diagonal matrices with 1 in the bottom right entry. We denote these two groups by U and D respectively. We let $U(\alpha)=\begin{pmatrix}1&\alpha\\0&1\end{pmatrix}$, and $D(t)=\begin{pmatrix}t&0\\0&1\end{pmatrix}$.

First, we give a basis of $V_{-\vec{t}-\vec{s},\vec{s}} \otimes V_{\vec{t'},\vec{s'}}$. We note that:

$$V_{-\vec{t}-\vec{s},\vec{s}} \otimes V_{\vec{t'},\vec{s'}} \cong \begin{pmatrix} (\det{}^{-s'_0} \otimes \mathrm{Sym}^{s'_0}\overline{\mathbb{F}}^2)^{Fr^{f-1}} \otimes \dots \otimes (\det{}^{-s'_{i-1}} \otimes \mathrm{Sym}^{s'_{i-1}}\overline{\mathbb{F}}^2)^{Fr^{f-i}} \\ \otimes(\det{}^{-s'_i-1} \otimes \mathrm{Sym}^{s'_i+2}\overline{\mathbb{F}}^2)^{Fr^{f-1-i}} \otimes (\det{}^{-s'_{i+1}} \otimes \mathrm{Sym}^{s'_{i+1}}\overline{\mathbb{F}}^2)^{Fr^{f-2-i}} \otimes \dots \\ \otimes(\det{}^{-s'_{f-1}} \otimes \mathrm{Sym}^{s'_{f-1}}\overline{\mathbb{F}}^2)^{Fr^0} \end{pmatrix}$$
$$\otimes \begin{pmatrix} (\det{}^0 \otimes \mathrm{Sym}^{s'_0}\overline{\mathbb{F}}^2)^{Fr^{f-1}} \otimes (\det{}^0 \otimes \mathrm{Sym}^{s'_1}\overline{\mathbb{F}}^2)^{Fr^{f-2}} \otimes \dots \\ \otimes(\det{}^0 \otimes \mathrm{Sym}^{s'_{f-1}}\overline{\mathbb{F}}^2)^{Fr^0} \end{pmatrix}$$

This can be viewed as a tensor of $2f$ terms, each term being a tensor of a determinant power and a symmetric power. The first f terms correspond to those coming from $V_{-\vec{t}-\vec{s},\vec{s}}$ and for each such term, a basis is given by homogeneous degree s_j monomials in variables w and z. Here, w corresponds to the first standard basis element of $\overline{\mathbb{F}}^2$, while z corresponds to the second standard basis element. The last f terms correspond to those coming from $V_{\vec{t'},\vec{s'}}$ and for each such term, a basis is given by homogeneous degree s'_j monomials in variables x and y. As before, x corresponds to the first standard basis element of $\overline{\mathbb{F}}^2$, while y corresponds to the second standard basis element.

Denote by W the $\overline{\mathbb{F}}$ subspace of $V_{-\vec{t}-\vec{s},\vec{s}} \otimes V_{\vec{t'},\vec{s'}}$ spanned by $\{(\otimes_{j=0}^{f-1} w^{k_j} z^{s_j-k_j}) \otimes y^{s'_0} \otimes y^{s'_1} \otimes \dots \otimes y^{s'_{f-1}}\}_{(k_j)_j}$. Evidently, W is a quotient as a $\langle U, D\rangle \subset \mathcal{K}$ representation. We now define a further partial order on the indexing set of the basis of W. Let $(k_j)_j$ and $(k'_j)_j$ be two indices, where $(k_j)_j$ corresponds to the basis element $(\otimes_{j=0}^{f-1} w^{k_j} z^{s_j-k_j}) \otimes y^{s'_0} \otimes y^{s'_1} \otimes \dots \otimes y^{s'_{f-1}}$ while $(k'_j)_j$ corresponds to the basis element $(\otimes_{j=0}^{f-1} w^{k'_j} z^{s_j-k'_j}) \otimes y^{s'_0} \otimes y^{s'_1} \otimes \dots \otimes y^{s'_{f-1}}$. If $k'_j \geq k_j$ for all j, then we say that $(k_j)_j$ is a descendant of $(k'_j)_j$. More precisely, if $\sum\limits_j (k'_j - k_j) = n \geq 0$, we say that $(k_j)_j$ is a n-descendant of $(k'_j)_j$. Alternatively, we say $(k'_j)_j$ is an n-ascendant of $(k_j)_j$, or $(k_j)_j$ is a $-n$-ascendant of $(k'_j)_j$, or $(k'_j)_j$ is a $-n$-descendant of $(k_j)_j$.

Now, take $\kappa_i^l A + \epsilon_i B + \kappa_i^u C$ to be the cocycle defined in Lemma 2.13. Denote by ψ the restriction of this cocycle to $\langle U, D\rangle \cap \mathcal{K}$. Then $\psi = \epsilon_i B + \kappa_i^u C$. Suppose it has an extension to $\langle U, D\rangle$. On composing the extension with the quotient map $V_{-\vec{t}-\vec{s},\vec{s}} \otimes V_{\vec{t'},\vec{s'}} \to W$, we obtain a cocycle valued in W, which we denote by q. Denote by $q_{(k_j)_j}$ the coordinates of q corresponding to the basis vector $(\otimes_{j=0}^{f-1} w^{k_j} z^{s_j-k_j}) \otimes y^{s'_0} \otimes y^{s'_1} \otimes \dots \otimes y^{s'_{f-1}}$. [Cautionary note about the notation: here the exponent of w is k_j, whereas in Lemma 2.13, $s_j - k_j$ is the exponent of w].

From the definition of ψ, $q_{(k_j)_j}|_{U\cap\mathcal{K}} \neq 0$ if and only if $k_j = s_j$ for all j. Further, $q_{(k_j)_j}|_{D\cap\mathcal{K}} \neq 0$ if and only if $k_i = s_i - 1 = s'_i + 1$ and $k_j = s_j = s'_j$ for all $j \neq i$. Each $(\otimes_{j=0}^{f-1} w^{k_j} z^{s_j-k_j}) \otimes y^{s'_0} \otimes y^{s'_1} \otimes \dots \otimes y^{s'_{f-1}} \in W$ is an eigenvector for $D(t)$ with eigenvalue $t^{\lambda_{(k_j)_j}}$, where $\lambda_{(k_j)_j}$ is the unique number in $[0, p^f - 1)$ that is equivalent to $\sum_{j\neq i} p^{f-1-j}(k_j - s'_j) + p^{f-1-i}(k_i - s'_i - 1) \bmod p^f - 1$. We make some observations about $\lambda_{(k_j)_j}$:

(1) $\lambda_{(k_j)_j} = 0$ if and only if $k_i = s'_i + 1$ and $k_j = s'_j$ for all $j \neq i$ if and only if $q_{(k_j)_j}|_{D\cap\mathcal{K}} \neq 0$.

(2) $\lambda_{(k_j)_j}$ are evidently pairwise distinct.

(3) Suppose $(k_j)_j \neq (s_j)_j$. Then $\lambda_{(k_j)_j} \neq p^{f-1-l}$ for any $l \in [0, f-1]$.

To see this, suppose on the contrary that $\mod p^f - 1$,

$$\sum_{j \neq i} p^{f-1-j}(k_j - s'_j) + p^{f-1-i}(k_i - s'_i - 1) = p^{f-1-l}.$$

Equivalently,

$$\sum_{j} p^{f-1-j} k_j \equiv \sum_{j \neq i} p^{f-1-j} s'_j + p^{f-1-i}(s'_i + 1) + p^{f-1-l}. \tag{2.14.1}$$

We have three subcases:

- If $l = i$, then the right hand side of (2.14.1) is $\sum_{j \in T - \{i\}} p^{f-1-j} s_j$. As each s_j is less than or equal to $p-1$, and at least one s_j is strictly less than $p-1$ (by assumption), k_j is forced to equal s_j for each j, a contradiction.
- If $l \neq i$ and $s'_l < p-1$, $s'_l + 1 \leq p-1$. Further, $s'_i + 1 < p-1$, because $s_i = s'_i + 2 \leq p-1$. Therefore, both the right and left hand sides have all coefficients of p^{f-1-j} less than or equal to $p-1$, and at least one coefficient strictly less than $p-1$, forcing right and left hand side coefficients to be the same. But this is a contradiction, since the coefficient of p^{f-1-l} clearly differs as for each j, $k_j \leq s_j$.
- If $l \neq i$ and $s'_l = p-1$, $s'_l + 1 = p$. By carrying over to obtain the coefficient of each p^{f-1-j} in the $[0, p-1]$ range, we see that the right hand side (2.14.1) is equivalent to a number with the coefficient of p^{f-1-m} equal to $s_m + 1$ for some $m \in [l-1, i+1]$. Thus both the right and left hand sides can be made to have all coefficients of p^{f-1-j} less than or equal to $p-1$, and at least one coefficient strictly less than $p-1$. Thus the coefficients on the two sides after carry must be the same. However, k_m is necessarily $\leq s_m$, giving a contradiction.

For each $(k_j)_j$, since D acts diagonally on W, $q_{(k_j)_j}|_D$ is a cocycle $D \to \overline{\mathbb{F}}(\lambda_{(k_j)_j})$, where $\overline{\mathbb{F}}(\lambda_{(k_j)_j})$ is a one-dimensional $\overline{\mathbb{F}}$-vector space with action of $D(t)$ given by multiplication with $t^{\lambda_{(k_j)_j}}$. Note that $D \cong \mathcal{O}_K^* \cong k^* \times (1 + \pi\mathcal{O}_K)$. Therefore, when $q_{(k_j)_j}|_{D \cap \mathcal{K}} = 0$, $q_{(k_j)_j}|_D$ can be seen as a cocycle $k^* \to \overline{\mathbb{F}}(\lambda_{(k_j)_j})$. For non-zero $\lambda_{(k_j)_j}$, $\sum_{\xi \in k^*} \xi^{\lambda_{(k_j)_j}} = 0$, because if $\tilde{\xi}$ is the generator of the cyclic group k^*,

$$\tilde{\xi}^{\lambda_{(k_j)_j}} \sum_{\xi \in k^*} \xi^{\lambda_{(k_j)_j}} = \sum_{\xi \in k^*} (\tilde{\xi}\xi)^{\lambda_{(k_j)_j}} = \sum_{\xi \in k^*} \xi^{\lambda_{(k_j)_j}}.$$

It follows that $H^1(k^*, \overline{\mathbb{F}}(\lambda_{(k_j)_j})) = \overline{\mathbb{F}}/(\tilde{\xi}^{\lambda_{(k_j)_j}} - 1)\overline{\mathbb{F}} = 0$. Therefore, when $q_{(k_j)_j}|_{D \cap \mathcal{K}} = 0$, there exists $a_{(k_j)_j} \in \overline{\mathbb{F}}$ such that $q_{(k_j)_j}(D(\xi)) = \xi^{\lambda_{(k_j)_j}} a_{(k_j)_j} - a_{(k_j)_j}$. When $q_{(k_j)_j}|_{D \cap \mathcal{K}} \neq 0$, let $a_{(k_j)_j} = 0$. Adjust the cocycle q by the coboundary given by the vector whose coordinate corresponding to $(\otimes_{j=0}^{f-1} w^{k_j} z^{s_j - k_j}) \otimes y^{s'_0} \otimes y^{s'_1} \otimes \cdots \otimes y^{s'_{f-1}}$ is $-a_{(k_j)_j}$. Therefore, we may assume that when $q_{(k_j)_j}|_{D \cap \mathcal{K}} = 0$, $q_{(k_j)_j}|_D = 0$. When $q_{(k_j)_j}|_{D \cap \mathcal{K}} \neq 0$, since $\lambda_{(k_j)_j} = 0$, $q_{(k_j)_j}|_D$ is a group homomorphism $k^* \times (1 + \pi\mathcal{O}_K) \cong D \to \overline{\mathbb{F}}$. Since order of k^* is prime to p, $q_{(k_j)_j}(D(k^*)) = 0$. Therefore, regardless of $(k_j)_j$, we have $q_{(k_j)_j}(D(k^*)) = 0$.

Our next order of business is to understand each $q_{(k_j)_j}|_U$. Note that $U \cong \mathcal{O}_K$. Except when $k_j = s_j$ for all j, $q_{(k_j)_j}|_{U(\pi\mathcal{O}_K)} = 0$ (as remarked earlier) and therefore, $q_{(k_j)_j}|_U$ can be seen as a map on $\mathcal{O}_K/\pi$. Since $D(\xi)U(\alpha) = U(\xi\alpha)D(\xi)$ for $\xi \in k^*$ and $\alpha \in \mathcal{O}_K/\pi$, we have the following for all $(k_j)_j \neq (s_j)_j$:

(2.14.2)
$$\xi^{\lambda_{(k_j)_j}} q_{(k_j)_j}(U(\alpha)) = q_{(k_j)_j}(D(\xi)U(\alpha)) = q_{(k_j)_j}(U(\xi\alpha)D(\xi)) = q_{(k_j)_j}(U(\xi\alpha))$$

Therefore, replacing α with 1 and ξ with α (this covers all the cases since $q_{(k_j)_j}(U(0))$ is already 0 because $q_{(k_j)_j}|_{U\cap\mathcal{K}} = 0$), we obtain for all $(k_j)_j \neq (s_j)_j$:

$$q_{(k_j)_j}(U(\alpha)) = \alpha^{\lambda_{(k_j)_j}} q_{(k_j)_j}(U(1)) \tag{2.14.3}$$

Now, we do an inductive argument to show that $q_{(k_j)_j}|_U = 0$ for all $(k_j)_j \neq (s_j)_j$.

Suppose $q_{(k_j)_j}|_U = 0$ for each m-ascendant $(k_j)_j$ of $(0)_j$, where $-1 \leq m < (\sum_j s_j) - 1$. The base case with $m = -1$ is automatic, because $(0)_j$ has no descendants. We will show that $q_{(k_j)_j}|_U = 0$ for each $m+1$-ascendant $(k_j)_j$ of $(0)_j$.

Fix an $m+1$-ascendant $(k_j)_j$ of $(0)_j$. Take $(k'_j)_j$ to be a 1-ascendant of $(k_j)_j$ (therefore, an $m+2$-ascendant of $(0)_j$).

Since q is a cocycle, we have for each $0 \neq \alpha \in \mathcal{O}_K/\pi$:

$$q_{(k'_j)_j}(U(\alpha)) + q_{(k'_j)_j}(U(1)) + \sum_{\substack{(l_j)_j \in 1\text{-descendants} \\ \text{of } (k'_j)_j}} \left(\left(\prod_j \binom{s_j - l_j}{s_j - k'_j} \right) \alpha^{\sum_j p^{f-1-j}(k'_j - l_j)} q_{(l_j)_j}(U(1)) \right)$$

$$= q_{(k'_j)_j}(U(\alpha+1))$$

$$= q_{(k'_j)_j}(U(1+\alpha))$$

$$= q_{(k'_j)_j}(U(1)) + q_{(k'_j)_j}(U(\alpha)) + \sum_{\substack{(l_j)_j \in 1\text{-descendants} \\ \text{of } (k'_j)_j}} \left(\left(\prod_j \binom{s_j - l_j}{s_j - k'_j} \right) q_{(l_j)_j}(U(\alpha)) \right)$$

Therefore,

$$\sum_{\substack{(l_j)_j\in 1\text{-descendants}\\ \text{of } (k'_j)_j}} \left(\left(\prod_j \binom{s_j-l_j}{s_j-k'_j}\right)\alpha^{\sum_j p^{f-1-j}(k'_j-l_j)} q_{(l_j)_j}(U(1))\right)$$

$$= \sum_{\substack{(l_j)_j\in 1\text{-descendants}\\ \text{of } (k'_j)_j}} \left(\left(\prod_j \binom{s_j-l_j}{s_j-k'_j}\right) q_{(l_j)_j}(U(\alpha))\right)$$

$$= \sum_{\substack{(l_j)_j\in 1\text{-descendants}\\ \text{of } (k'_j)_j}} \left(\left(\prod_j \binom{s_j-l_j}{s_j-k'_j}\right) \alpha^{\lambda_{(l_j)_j}} q_{(l_j)_j}(U(1))\right) \qquad \text{(by (2.14.3))}$$

Therefore, each $\alpha \in k^*$ satisfies the following polynomial in x:

$$\sum_{\substack{(l_j)_j\in 1\text{-descendants}\\ \text{of } (k'_j)_j}} \left(\prod_j \binom{s_j-l_j}{s_j-k'_j}\right) q_{(l_j)_j}(U(1))x^{\lambda_{(l_j)_j}}$$

$$- \sum_{\substack{(l_j)_j\in 1\text{-descendants}\\ \text{of } (k'_j)_j}} \left(\prod_j \binom{s_j-l_j}{s_j-k'_j}\right) q_{(l_j)_j}(U(1))x^{\sum_j p^{f-1-j}(k'_j-l_j)}$$

If non-zero, this polynomial is of degree less than $p^f - 1$, with at least $|k^*|$ distinct roots, a contradiction. Note that $\sum_j p^{f-1-j}(k'_j - l_j) = p^{f-1-m_{(l_j)}}$ for some $m_{(l_j)} \in [0, f-1]$.

Since $\lambda_{(k_j)_j}$ does not equal any of the $\sum_j p^{f-1-j}(k'_j - l_j)$ terms, the coefficient of $x^{\lambda_{(k_j)_j}}$ is $\left(\prod_j \binom{s_j-k_j}{s_j-k'_j}\right) q_{(k_j)_j}(U(1))$ and it must equal 0. This implies that $q_{(k_j)_j}|_U = 0$ by (2.14.3).

Finally, we come to the last leg of the proof. Because $q_{(k_j)_j}|_U = 0$ for each $(k_j)_j \neq (s_j)_j$, $q_{(s_j)_j}(U(\alpha+\beta)) = q_{(s_j)_j}(U(\alpha)) + q_{(s_j)_j}(U(\beta))$. Therefore $q_{(s_j)_j}(U(p)) = pq_{(s_j)_j}(U(1)) = 0$. However, $q_{(s_j)_j}|_{U\cap\mathcal{K}} = \kappa_i^u|_{U\cap\mathcal{K}}$ (from the definition of C in Lemma 2.13). As p is the uniformizer of $\mathcal{O}_K$, $\kappa_i^u(U(p)) \neq 0$, giving a contradiction. □

3. Stack dimensions and extensions of G_K characters

Let σ and τ be a pair of non-Steinberg, non-isomorphic Serre weights. We record a fact from [DDR] and [Ste] that we will use in this section. Suppose χ_1 and χ_2 are distinct G_K characters such that the subspace of $\mathrm{Ext}^1_{\mathbb{F}[\mathrm{G}_K]}(\chi_2, \chi_1)$ corresponding to representations with Serre weights both σ and τ has dimension d. Suppose χ'_1 and χ'_2 are unramified twists of χ_1 and χ_2 respectively. If $\chi'_1 \neq \chi'_2$, then the subspace of $\mathrm{Ext}^1_{\mathbb{F}[\mathrm{G}_K]}(\chi'_2, \chi'_1)$ that corresponds to representations with Serre weights σ and τ also has dimension d. If on the other hand, $\chi'_1 = \chi'_2$, then a $(d+1)$-dimensional subspace of $\mathrm{Ext}^1_{\mathbb{F}[\mathrm{G}_K]}(\chi_2', \chi_1')$ corresponds to representations with Serre weights σ and τ.

We now make a few definitions, before stating our main propositions relating dimensions of closed substacks of $\mathcal{X}$ to vector space dimensions of extensions of G_K characters.

Definition 3.1. Let χ_1 and χ_2 be a pair of $\mathbb{F}$-valued G_K characters. We say that a set $\mathcal{F}_{\chi_1,\chi_2}$ of G_K-representations with $\overline{\mathbb{F}}$-coefficients is a *family of representations* if each representation in $\mathcal{F}_{\chi_1,\chi_2}$ is an extension of an unramified twist of χ_2 by an unramified twist of χ_1.

Definition 3.2. Consider $\mathbb{G}_m \times \mathbb{G}_m$ as parametrizing the unramified twists of χ_1 and χ_2 via the value of the unramified characters on Frob_K. We say that the family $\mathcal{F}_{\chi_1,\chi_2}$ is *of dimension* $\leq d$ (resp. *of dimension d)* if there exists a dense open subset W of $\mathbb{G}_m \times \mathbb{G}_m$ such that the following condition is satisfied: if χ_1' and χ_2' are unramified twists of χ_1 and χ_2 (respectively) corresponding to an $\overline{\mathbb{F}}$-point of W, then the extensions in $\mathrm{Ext}^1_{\mathbb{F}[\mathrm{G}_K]}(\chi_2', \chi_1')$ giving elements of $\mathcal{F}_{\chi_1,\chi_2}$ form a subspace of dimension $\leq d$ (resp. of dimension d).

Definition 3.3. We say that two families $\mathcal{F}_{\chi_1,\chi_2}$ and $\mathcal{F}_{\chi_1',\chi_2'}$ are *separated* if χ_1' and χ_2' are not both unramified twists of χ_1 and χ_2 respectively.

We now recall some constructions from [EG2, Sec. 5] before stating our main propositions. We note first that there exist finitely many $\overline{\mathbb{F}}$-valued characters of I_K that admit extensions to G_K. Each such character is in fact valued in $\mathbb{F}$ and is described uniquely by $a = (a_i)_{i\in T}$ with each $a_i \in [0, p-1]$ and at least some $a_i < p-1$. Let A be the set of such f tuples. Then for $a \in A$, the corresponding I_K character is given by $\prod_{i\in T} \omega_i^{a_i}|_{\mathrm{I}_K}$. Fix an extension of such a character to G_K, and denote it by ψ_a. When each $a_i = 0$, take $\psi_a = 1$. When each $a_i = e$ and $p > 2$, take ψ_a to be the mod p cyclotomic character, ϵ.

Let M be the rank 1 (φ, Γ)-module over $\mathbb{G}_m := \mathrm{Spec}\ \mathbb{F}[x, x^{-1}]$ generated by some $v \in M$ such that $\varphi(v) = xv$ and Γ action is trivial. By applying the functor $\mathbf{D}$ defined in [EG1, Sec. 3.6], we obtain a set of (φ, Γ)-modules $\{\mathbf{D}(\psi_a)\}_{a\in A}$ defined over $\mathbb{F}$. Let M_a denote the (φ, Γ)-module $\mathbf{D}(\psi_a) \boxtimes M$ defined over $\mathbb{G}_m$. For a subscheme $\mathrm{Spec}\ R \subset \mathbb{G}_m$, we will denote by $M_a|_{\mathrm{Spec}\ R}$ the (φ, Γ)-module obtained by changing scalars to R. Let $X_a = \mathbb{G}_m$ when ψ_a is not trivial or cyclotomic, and let $X_a = \mathbb{G}_m \setminus \{1\}$ when ψ_a is trivial or cyclotomic.

The cohomology groups of $\mathcal{C}^\bullet(M_a|_{X_a})$, the Herr complex associated to $M_a|_{X_a}$ by [EG1, Sec. 5], vanish in degrees 0 and 2 (the latter by Tate local duality). Therefore, as the cohomology group in degree 1 gives a coherent sheaf on X_a of constant rank $[K : \mathbb{Q}_p]$ by the local Euler characteristic formula, it is a locally free sheaf. Denoting the total space of this sheaf by V_a, we obtain a space parameterizing extensions of $\mathbf{D}(1)$ by $M_a|_{X_a}$, the former viewed as a (φ, Γ)-module over X_a by extension of scalars from $\mathbb{F}$ to the global sections on X_a. Thus, for each $b \in A$, twisting further by M_b, there exists a map

$$f_{a,b} : V_a \times \mathbb{G}_m \to \mathcal{X}$$

corresponding to the universal extension of $\mathbf{D}(1) \boxtimes M_b$ by $M_a|_{X_a} \boxtimes M_b$. Note that $\mathbb{G}_m \times \mathbb{G}_m$ acts on extensions of $\mathbf{D}(1) \boxtimes M_b$ by $M_a|_{X_a} \boxtimes M_b$, for e.g. as described in [EG2, Sec. 7.3]. Since $\mathbf{D}(1) \neq M_a|_{X_a}$, the induced map

$$\overline{f}_{a,b} : (V_a \times \mathbb{G}_m)/(\mathbb{G}_m \times \mathbb{G}_m) \to \mathcal{X}$$

is a monomorphism. There exists a map

$$\pi_{a,b} : V_a \times \mathbb{G}_m \to B_{a,b} := X_a \times \mathbb{G}_m$$

induced by the structure map $V_a \to X_a$ and the identity map $\mathbb{G}_m \to \mathbb{G}_m$. The map $\pi_{a,b}$ corresponds to choices of unramified twists of $\mathbf{D}(\psi_a)$ and $\mathbf{D}(\psi_b)$ respectively.

Now, we consider the Herr complex associated to $\mathbf{D}(1)$, denoted $\mathcal{C}^\bullet(\mathbf{D}(1))$. Each of the cohomology groups is a finite dimensional vector space over $\mathbb{F}$. The considerations in [EG2, Sec. 5.4] show that for any $\mathbb{F}$-algebra R, $H^1(\mathcal{C}^\bullet(\mathbf{D}(1)_R)) = H^1(\mathcal{C}^\bullet(\mathbf{D}(1)) \otimes R$, where $\mathbf{D}(1)_R$ is the (φ, Γ)-module obtained from $\mathbf{D}(1)$ by extending the scalars to R. Therefore, the total space of the invertible sheaf on $\operatorname{Spec} \mathbb{F}$ corresponding to $H^1(\mathcal{C}^\bullet(\mathbf{D}(1))$ parameterizes extensions of $\mathbf{D}(1)$ by itself. Thus, denoting this total space by V_1, we can define a map

$$f_{1,b} : V_1 \times \mathbb{G}_m \to \mathcal{X}$$

giving the universal extension of $\mathbf{D}(1) \boxtimes M_b$ by $\mathbf{D}(1) \boxtimes M_b$. In this case, in addition to $\mathbb{G}_m \times \mathbb{G}_m$, there is an action of the upper unipotent group U on extensions of $\mathbf{D}(1) \boxtimes M_b$ by $\mathbf{D}(1) \boxtimes M_b$, thus giving a map

$$\overline{f}_{1,b} : (V_1 \times \mathbb{G}_m)/(\mathbb{G}_m \times \mathbb{G}_m \times U) \to \mathcal{X}.$$

Denote by $\pi_{1,b}$ the projection of $V_1 \times \mathbb{G}_m$ onto the second factor.

Finally, when $p > 2$, we consider the Herr complex associated to $\mathbf{D}(\epsilon)$, denoted $\mathcal{C}^\bullet(\mathbf{D}(\epsilon))$. As before, viewing the finite dimensional degree 1 cohomology group as an invertible sheaf on a point, the total space gives a vector bundle V_ϵ defined over $\operatorname{Spec} \mathbb{F}$ that parameterizes extensions of $\mathbf{D}(1)$ by $\mathbf{D}(\epsilon)$. Thus, we have a map

$$f_{\epsilon,b} : V_\epsilon \times \mathbb{G}_m \to \mathcal{X}$$

giving the universal extension of $\mathbf{D}(1) \boxtimes M_b$ by $\mathbf{D}(\epsilon) \boxtimes M_b$. The induced map

$$\overline{f}_{\epsilon,b} : (V_\epsilon \times \mathbb{G}_m)/(\mathbb{G}_m \times \mathbb{G}_m) \to \mathcal{X}$$

is a monomorphism. Denote by $\pi_{\epsilon,b}$ the projection of $V_\epsilon \times \mathbb{G}_m$ onto the second factor.

By construction, each finite type point of $\mathcal{X}$ corresponding to a reducible representation is in the image of one of the (finitely many) $f_{a,b}$, $f_{1,b}$ and $f_{\epsilon,b}$ maps.

Now, consider the set of isomorphism classes of irreducible 2-dimensional representations defined over $\mathbb{F}$. For each such representation $\overline{\rho}$, there exists a map $\operatorname{Spec} \mathbb{F} \to \mathcal{X}$, which in turn can be used to write a map $f_{\overline{\rho}} : \mathbb{G}_m \to \mathcal{X}$ corresponding to $\mathbf{D}(\overline{\rho}) \boxtimes M$. The finite type points in the image correspond to all the unramified twists of $\overline{\rho}$. Since the automorphisms of irreducible representations are precisely the invertible scalars, $f_{\overline{\rho}}$ factors via the monomorphism

$$\overline{f}_{\overline{\rho}} : [\mathbb{G}_m/\mathbb{G}_m] \to \mathcal{X}.$$

Since there are only finitely many isomorphism classes of irreducible 2-dimensional representations upto unramified twists, the finite type points of $\mathcal{X}$ corresponding to irreducible representations lie in the image of one of $f_{\overline{\rho}}$ for finitely many $\overline{\rho}$.

Fix non-Steinberg Serre weights σ and τ. Denoting by $\mathcal{E}$ the intersection of $\mathcal{X}_\sigma$ with $\mathcal{X}_\tau$, for each $a \in A$, let $Y_{a,b} := \mathcal{E} \times_{\mathcal{X}_2} V_{a,b}$. We also define $Y_{1,b} := \mathcal{E} \times_{\mathcal{X}_2} V_{1,b}$ and $Y_{\epsilon,b} := \mathcal{E} \times_{\mathcal{X}_2} V_{\epsilon,b}$. The maps $\pi_{a,b}|_{Y_{a,b}}$, $\pi_{1,b}|_{Y_{1,b}}$ and $\pi_{\epsilon,b}|_{Y_{\epsilon,b}}$ will henceforth be written simply as $\pi_{a,b}$, $\pi_{1,b}$ and $\pi_{\epsilon,b}$.

Proposition 3.4. *Let $d \geq 0$. Suppose all families of representations contained in $\mathcal{E}(\overline{\mathbb{F}})$ are of dimension $\leq d$, and moreover, $\mathcal{E}(\overline{\mathbb{F}})$ contains at least one family of dimension d. Then the following are true:*

(i) *$\mathcal{E}$ has dimension d.*
(ii) *If $d > 0$, the number of d-dimensional components in $\mathcal{E}$ equals the number of d-dimensional pairwise separated families contained in $\mathcal{E}$.*
(iii) *Let $d = 0$, and let*

$$C := \{\overline{\rho} : \mathrm{G}_K \to \mathrm{GL}_2(\mathbb{F}) \mid \overline{\rho} \text{ is semisimple}\}/\sim$$

where $\overline{\rho} \sim \overline{\rho}'$ if $\overline{\rho}$ and $\overline{\rho}'$ are isomorphic as I_K representations. Then the number of d-dimensional components in $\mathcal{E}$ equals $|C|$.

The proof of this proposition will use the following lemmas.

Lemma 3.5. *Let χ_1 and χ_2 be fixed G_K characters. Suppose $\mathcal{E}(\overline{\mathbb{F}})$ contains a family $\mathcal{F}_{\chi_1,\chi_2}$ of representations of dimension d. Suppose moreover that there is no other family of extensions of χ_2 by χ_1 contained in $\mathcal{E}(\overline{\mathbb{F}})$ with dimension $> d$. Let $a, b \in A$ be such that ψ_b is an unramified twist of χ_2, while $\psi_a \otimes \psi_b$ is an unramified twist of χ_1. Then the following are true:*

(i) *The dimension of the scheme-theoretic image of $Y_{a,b}$ is $\leq d$.*
(ii) *The number of d-dimensional components in the scheme-theoretic image of $Y_{a,b}$ is at most 1.*

Proof. (i) Let q be a closed point of $B_{a,b}$, and after fixing an embedding $\kappa(q) \hookrightarrow \overline{\mathbb{F}}$, let $\overline{q}$ be the corresponding $\overline{\mathbb{F}}$-point of $B_{a,b}$. By the construction of $B_{a,b}$, representations coming from $Y_{a,b}(\overline{\mathbb{F}})$ are never an extension of a character by itself. Therefore, the hypotheses in the statement of the Lemma force $\pi_{a,b}^{-1}(\overline{q})(\overline{\mathbb{F}})$ to be a vector space of dimension $\leq d$. We have:

Since the $\overline{\mathbb{F}}$-points of $\pi_{a,b}^{-1}(\overline{q})$ form a vector space, the reduced induced closed subscheme of $\pi_{a,b}^{-1}(q)$ must be cut out by homogeneous linear equations in $V_{a,b} \times \mathbb{G}_m \times \kappa(q)$ and thus be irreducible of dimension equal to the $\overline{\mathbb{F}}$-vector space dimension of $\pi_{a,b}^{-1}(\overline{q})(\overline{\mathbb{F}})$.

Let S be an irreducible component of $Y_{a,b}$. Denote by $\overline{f_{a,b}(S)}$ the scheme-theoretic image of S. By [Sta, Tag 0DS4], there exists a dense set $U \subset S$ such that for any $p \in U(\overline{\mathbb{F}})$, the dimension of $\overline{f_{a,b}(S)}$ is given by:

$$\dim \overline{f_{a,b}(S)} = \dim S - \dim_p(S_{f_{a,b}(p)}) = \dim_p S - \dim_p(S_{f_{a,b}(p)})$$

where,

$$\dim_p S \leq \dim \pi_{a,b}^{-1}(\pi_{a,b}(p)) + \dim\,(\overline{\pi_{a,b}(S)})$$

Restrict U further if necessary so that it is disjoint from other irreducible components of Y_j. Then for p a closed point in U, since $\pi_{a,b}^{-1}(\pi_{a,b}(p))$ is irreducible, it is contained entirely in some irreducible component of $Y_{a,b}$. By the conditions on U, $\pi_{a,b}^{-1}(\pi_{a,b}(p)) \subset S$. Therefore, $\dim_p(S_{f_{a,b}(p)}) = \dim_p(Y_{a,b})_{f_{a,b}(p)}$. Since $f_{a,b}|_{Y_{a,b}}$ factors through the quotient $Y_{a,b}/(\mathbb{G}_m \times \mathbb{G}_m)$, we obtain:

$$\dim_p(S_{f_{a,b}(p)}) = \dim_p(Y_{a,b})_{f_{a,b}(p)} = 2$$

Therefore the dimension of scheme-theoretic image of S is $\leq d - (2 - \dim\,(\overline{\pi_{a,b}(S)})) \leq d$.

(ii) For $i \in \{1,2\}$, suppose S^i is an irreducible component of $Y_{a,b}$ with a scheme-theoretic image of dimension d. Let U^i be the dense open subset of S^i obtained by taking the complement of all other irreducible components of $Y_{a,b}$. Therefore, $\overline{\pi_{a,b}(U^i)} = \overline{\pi_{a,b}(S^i)} = B_{a,b}$. Since $\pi_{a,b}(U^i)$ is constructible, it contains a dense open W^i of $B_{a,b}$. Let $W = W^1 \cap W^2$. If q is a closed point of W, $\pi_{a,b}^{-1}(q)$ is irreducible and contained entirely in at least one irreducible component of $Y_{a,b}$. But since for each i, $\pi_{a,b}^{-1}(q) \cap U^i$ is non-empty and disjoint from all irreducible components of $Y_{a,b}$ apart from S^i, S^1 must be the same as S^2. This shows that at most one irreducible component of $Y_{a,b}$ can have a d-dimensional scheme-theoretic image.

□

Lemma 3.6. *For each $b \in A$, the scheme theoretic images of $f_{\epsilon,b}$ and of $f_{1,b}$ are strictly less than d.*

Proof. The proof follows the same ideas as the proof of Lemma 3.5. The reduction in dimension for the scheme-theoretic image of $f_{1,b}$ arises from the fact that $\pi_{1,b} : V_1 \times \mathbb{G}_m \to \mathbb{G}_m$ has dimension 1 less than the target of $\pi_{a,b}$ along with the fact that $f_{1,b}$ factors through $(V_1 \times \mathbb{G}_m)/(\mathbb{G}_m \times \mathbb{G}_m \times U)$. When $p > 2$, the reduction in dimension for the scheme-theoretic image of $f_{\epsilon,b}$ arises from the fact that the target of the map $\pi_{\epsilon,b} : V_\epsilon \times \mathbb{G}_m \to \mathbb{G}_m$ has dimension 1 less than the target of $\pi_{a,b}$. □

Lemma 3.7. *Let χ_1 and χ_2 be fixed G_K characters. Suppose $\mathcal{E}(\overline{\mathbb{F}})$ contains a family $\mathcal{F}_{\chi_1,\chi_2}$ of representations of dimension d. Suppose moreover that there is no other family of extensions of χ_2 by χ_1 contained in $\mathcal{E}(\overline{\mathbb{F}})$ with dimension $> d$. Let $a, b \in A$ be such that ψ_b is an unramified twist of χ_2, while $\psi_a \otimes \psi_b$ is an unramified twist of χ_1. Then the scheme-theoretic image of $Y_{a,b}$ has dimension d.*

Proof. By the construction of $B_{a,b}$, representations coming from $Y_{a,b}(\overline{\mathbb{F}})$ are never extensions of a character by itself. Therefore, for each unramified twist of χ_1 and χ_2 coming from twisting $\psi_a \otimes \psi_b$ and ψ_b by unramified characters corresponding to $\overline{\mathbb{F}}$-points of $B_{a,b}$, the space of extensions giving representations contained in $\mathcal{E}(\overline{\mathbb{F}})$ is precisely d-dimensional.

As $\pi_{a,b}$ is of finite type over an integral scheme, there exists a dense open W of $B_{a,b}$ such that over W, $\pi_{a,b}$ is flat.

Let q be a closed point of W. Fix an embedding of $\kappa(q)$ into $\overline{\mathbb{F}}$ to view q as a $\overline{\mathbb{F}}$-point $\overline{q}$. By hypothesis, $(Y_{a,b})_{\overline{q}}(\overline{\mathbb{F}})$ has the structure of a $\overline{\mathbb{F}}$-vector space of dimension d. Therefore, $(Y_{a,b})_{\overline{q}}$ (and hence $(Y_{a,b})_q$) is irreducible of dimension d.

By flatness over W, $\dim Y_{a,b}|_W = \dim\, (Y_{a,b})_q + 2 = d + 2$. Therefore, there exists an irreducible component S of $Y_{a,b}|_W$ with dimension $d+2$. Denote by $\overline{f_{a,b}(S)}$ the scheme-theoretic image of S. As in the proof of Lemma 3.5, there exists a dense open subset U of S, such that for all $p \in U$,

$$\dim \overline{f_{a,b}(S)} = \dim S - \dim_p(S_{f(p)})$$

and

$$\dim_p(S_{f_{a,b}(p)}) = 2$$

Therefore, the dimension of $\overline{f_{a,b}(S)}$, and of $Y_{a,b}$, is precisely d (it cannot exceed d by Lemma 3.5). □

Proof of Proposition 3.4. Recall that each reducible representation is in the literal image of either $f_{a,b}$ or $f_{1,b}$ or $f_{1,b}$ for some $a, b \in A$. Moreover, irreducible representations contribute to finitely many zero-dimensional substacks of $\mathcal{E}$ by the description of the maps $\overline{f}_{\overline{\rho}}$ for ρ irreducible.

Therefore, the first statement follows from Lemmas 3.5 and 3.6, which also show that each d-dimensional family contains precisely one d-dimensional component in its closure.

Now assume that $\mathcal{Y}$ is a top dimensional component in $\mathcal{E}$ contained in the closure of two separated d-dimensional families $\mathcal{F}_{\chi_1,\chi_2}$ and $\mathcal{F}_{\chi_1',\chi_2'}$. Then there exist unique $a, a', b, b' \in A$ so that (ψ_a, ψ_b) and $(\psi_{a'}, \psi_{b'})$ are unramified twists of (χ_1, χ_2) and (χ_1', χ_2') respectively. Therefore, $\mathcal{Y}$ is in the scheme-theoretic image of both $Y_{a,b}$ and $Y_{a',b'}$ (this uses Lemma 3.6 which shows that the scheme-theoretic image of $Y_{1,b}$ and $Y_{\epsilon,b}$ is necessarily of dimension less than d).

Let W (resp. W') be a dense open subset of $B_{a,b}$ (resp. $B_{a',b'}$) for each $\overline{q} \in B_{a,b}(\overline{\mathbb{F}})$ (resp. $\overline{q} \in B_{a',b'}(\overline{\mathbb{F}})$), each $\overline{\mathbb{F}}$ point of $(Y_{a,b})_{\overline{q}}$ (resp. $(Y_{a',b'})_{\overline{q}}$) corresponds to a representation contained in $\mathcal{F}_{\chi_1,\chi_2}$ (resp. $\mathcal{F}_{\chi_1',\chi_2'}$).

By the arguments in Lemma 3.7, $\mathcal{Y}$ is contained in the scheme-theoretic image of an irreducible component S (resp. S') of $Y_{a,b}|_W$ (resp. $Y'_{a,b}|_{W'}$). Since the images of $|S|$ and of $|S'|$ in $|\mathcal{Y}|$ are constructible sets dense in $\mathcal{Y}$, there exists a dense open U of $|\mathcal{Y}|$ contained in both $|S|$ and $|S'|$. If $(a, b) \neq (a', b')$, then this means that $(a, b) = (b', a')$ and U contains only split extensions. Therefore, families of split extensions are dense in $\mathcal{Y}$. If $d > 0$, this is an impossibility because the split locus correspond to a dimension 2 closed substack of $Y_{a,b}$ whose scheme-theoretic image has dimension 0 by the arguments in Lemmas 3.5 and 3.7. This settles the second statement, and along with the fact that the image of each $\overline{f}_{\overline{\rho}}$ is 0-dimensional, settles the third statement as well. □

4. Computations of Serre weights

4.1. Linear algebraic formulation for Serre weights of G_K-representations. In the subsequent text, we will write our f-tuples with decreasing indices. We recall some relevant results from [Ste] and [DDR] below. Let $\overline{\rho}$ be a G_K representation of the form $\begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix} \in \mathrm{Ext}^1_{\mathrm{G}_K}(\overline{\mathbb{F}}(\chi_2), \overline{\mathbb{F}}(\chi_1))$.

$V_{\vec{t},\vec{s}}$ is a Serre weight of $\overline{\rho}^\vee$ if and only if $V_{\vec{t},\vec{s}}$ is a Serre weight of $\overline{\rho}$ in the sense of [Ste] and [DDR]. Thus $V_{\vec{t},\vec{s}}$ is a Serre weight of $\overline{\rho}^\vee$ iff the following conditions are met:

(1) There exists a subset J of T, and for each $i \in T$ there exists $x_i \in [0, e-1]$ such that:

$$\chi_1|_{I_K} = \prod_{i \in T} \omega_i^{t_i} \prod_{i \in J} \omega_i^{s_i+1+x_i} \prod_{i \in J^c} \omega_i^{x_i} \tag{4.1.1}$$

and

$$\chi_2|_{I_K} = \prod_{i \in T} \omega_i^{t_i} \prod_{i \in J} \omega_i^{e-1-x_i} \prod_{i \in J^c} \omega_i^{s_i+e-x_i} \tag{4.1.2}$$

(2) $\overline{\rho} \in L_{V_{\vec{t},\vec{s}}}(\overline{\mathbb{F}}(\chi_1), \overline{\mathbb{F}}(\chi_2)) \subset \mathrm{Ext}^1_{\mathrm{G}_K}(\overline{\mathbb{F}}(\chi_2), \overline{\mathbb{F}}(\chi_1))$, where $L_{V_{\vec{t},\vec{s}}}(\overline{\mathbb{F}}(\chi_1), \overline{\mathbb{F}}(\chi_2))$ (or just $L_{V_{\vec{t},\vec{s}}}$ if χ_1 and χ_2 are understood) is a particular distinguished subspace.

Assuming (4.1.1) and (4.1.2), we now note the recipe for obtaining $L_{V_{\vec{t},\vec{s}}}$ as given in [Ste], with slight differences in notation.

We first write $\chi_2|_{I_K} = \prod_{i\in T}\omega_i^{t_i}\prod_{i\in T}\omega_i^{m_i}$ for the unique $m_i \in [0, p-1]$ with not all m_i equal to $p-1$. Let $\mathcal{S}$ be the set of f-tuples of non-negative integers $(a_{f-1}, a_{f-2}, ..., a_0)$ satisfying $\chi_2|_{I_K} = \prod_{i\in T}\omega_i^{t_i}\prod_{i\in T}\omega_i^{a_i}$ and $a_i \in [0, e-1] \cup [s_i + 1, s_i + e]$ for all i. Evidently, $\mathcal{S}$ is non-empty.

For $i \neq f-1$, let v_i be the f-tuple $(0, ..., 0, p, -1, 0, ..., 0)$ with -1 in i position, p in $i+1$ position and 0 everywhere else. Let v_{f-1} be $(-1, 0, ..., 0, p)$. Then there exists a subset $A \subset T$ such that

$$(m_{f-1}, ..., m_0) + \sum_{i\in A} v_i \in \mathcal{S} \tag{4.1.3}$$

Definition 4.2. Define $A_{\min}$ to be the minimal A satisfying (4.1.3), in the sense that it is contained in any other subset of T satisfying (4.1.3).

Definition 4.3. Given $(m_{f-1}, ..., m_0)$ and $A_{\min}$ as above.

$$(y_{f-1}, ..., y_0) := (m_{f-1}, ..., m_0) + \sum_{i\in A_{\min}} v_i \in \mathcal{S} \tag{4.3.1}$$

$$z_i := s_i + e - y_i \text{ for all } i \tag{4.3.2}$$

The indices of y_i's and z_i's will be interpreted to be elements of $\mathbb{Z}/f\mathbb{Z}$.

Remark 4.4. $\chi_1 = \prod_{i\in T}\omega_i^{z_i}\prod_{i\in T}\omega_i^{t_i}$, $\chi_2 = \prod_{i\in T}\omega_i^{y_i}\prod_{i\in T}\omega_i^{t_i}$ and $\chi_2^{-1}\chi_1 = \prod_{i\in T}\omega_i^{z_i - y_i}$.

Definition 4.5. If $y_i \geq s_i + 1$, let $\mathcal{I}_i := [0, z_i - 1]$, and if $y_i < s_i + 1$, let $\mathcal{I}_i := \{y_i\} \cup [s_i + 1, z_i - 1]$. Here the interval $[0, z_i - 1]$ is interpreted as the empty set if $z_i - 1 < 0$. We follow similar convention for $[s_i + 1, z_i - 1]$ when $z_i - 1 < s_i + 1$.

Remark 4.6. When $e = 1$, $\mathcal{I}_i = \{0\}$ if $y_i = 0$ and $\mathcal{I}_i = \emptyset$ if $y_i = s_i + 1$.

Remark 4.7. If $y_i \geq s_i + 1$, then $|\mathcal{I}_i| \leq e - 1$ with equality if and only if $y_i = s_i + 1$. If $y_i < s_i + 1$, then since $z_i \leq s_i + e$, $|\mathcal{I}_i| \leq e$ with equality if and only if $z_i = s_i + e$ or equivalently, $y_i = 0$.

Suppose $\chi_2^{-1}\chi_1 = \prod_{i\in T}\omega_i^{a_i}$ for $a_i \in [1, p]$ and not all $a_i = p$. We will extend the indices of the a_i to all of $\mathbb{Z}$ by setting $a_j = a_{j'}$ if $j \equiv j' \mod f$. We call the tuple $(a_{f-1}, ..., a_0)$ the *tame signature* of $\chi_2^{-1}\chi_1$. $\mathrm{Gal}(k/\mathbb{F}_p) = \langle \mathrm{Frob} \rangle \cong \mathbb{Z}/f\mathbb{Z}$ acts on such tuples $(a_{f-1}, ..., a_0)$ via

$$\mathrm{Frob} \cdot (a_{f-1}, ..., a_0) = (a_0, a_{f-1}, ..., a_1) \tag{4.7.1}$$

Let f' be the cardinality of the orbit of $(a_{f-1}, ..., a_0)$ under the action of $\mathrm{Gal}(k/\mathbb{F}_p)$, and let $f'' := f/f'$.

Definition 4.8. Let $n_i \in [0, p^f - 1)$ be such that $\chi_2^{-1}\chi_1|_{I_K} = \omega_i^{n_i}|_{I_K}$.

Note that $n_i = n_j$ iff $i \equiv j \mod f'$.

Definition 4.9. For $i \in T$, let

$$\lambda_i := \sum_{j=0}^{f-1} (z_{i+j+1} - y_{i+j+1}) p^{f-1-j} \tag{4.9.1}$$

$$\xi_i := (p^f - 1) z_i + \lambda_i \tag{4.9.2}$$

Definition 4.10. Let $J^{AH}_{V_{\vec{t},\vec{s}}}(\chi_1, \chi_2)$ denote the subset of all $\alpha = (m, \kappa) \in \mathbb{Z} \times \{0, ..., f'' - 1\}$ satisfying:

(i) $\exists i \in T$ and $u \in \mathcal{I}_i$, such that if ν is the p-adic valuation of $\xi_i - u(p^f - 1)$, then

$$m = \frac{\xi_i - u(p^f - 1)}{p^\nu} \tag{4.10.1}$$

(ii) Let $i_m \in \{0, ..., f' - 1\}$ be such that $m \equiv n_{i_m} \mod p^f - 1$. It exists because by the above, $p^\nu m \equiv n_i \mod p^f - 1$, and so, $m \equiv n_{i-\nu}$. We require that κ satisfies

$$i_m + \kappa f' \equiv i - \nu \mod f \tag{4.10.2}$$

By [Ste, Prop. 3.13], for each $i \in T$ and $u \in \mathcal{I}_i$, there exists a unique α satisfying the conditions above. By [Ste, Thm. 3.16], each α in $J^{AH}_{V_{\vec{t},\vec{s}}}(\chi_1, \chi_2)$ gives a unique basis element of $L_{V_{\vec{t},\vec{s}}}(\chi_1, \chi_2)$, denoted as c_α. $L_{V_{\vec{t},\vec{s}}}(\chi_1, \chi_2)$ is the span of c_α's together with additional, distinguished basis elements c_{un} if $\chi_2^{-1}\chi_1$ is trivial and c_{tr} if $\chi_2^{-1}\chi_1$ is cyclotomic, $\prod_{i \in T} \omega^{-t_i} \otimes \chi_2$ is unramified and $s_i = p - 1$ for all i. A consequence of these results is that

$$\dim_{\overline{\mathbb{F}}} L_{V_{\vec{t},\vec{s}}}(\chi_1, \chi_2) = \sum_{i \in T} |\mathcal{I}_i| + \delta \tag{4.10.3}$$

where δ depends on the situation and could be 0 or 1 for $p > 2$, and 0, 1 or 2 for $p = 2$. It is always 0 if $\chi_2^{-1}\chi_1$ is neither trivial nor cyclotomic.

Consider two Serre weights $V_{\vec{t},\vec{s}}$ and $V_{\vec{t'},\vec{s'}}$. Suppose there exist subsets J and J' of T, and for each $i \in T$, there exist $x_i, x'_i \in [0, e-1]$ such that:

$$\chi_1|_{I_K} = \prod_{i \in T} \omega_i^{t_i} \prod_{i \in J} \omega_i^{s_i+1+x_i} \prod_{i \in J^c} \omega_i^{x_i} = \prod_{i \in T} \omega_i^{t'_i} \prod_{i \in J'} \omega_i^{s'_i+1+x'_i} \prod_{i \in J'^c} \omega_i^{x'_i} \tag{4.10.4}$$

and

$$\chi_2|_{I_K} = \prod_{i \in T} \omega_i^{t_i} \prod_{i \in J} \omega_i^{e-1-x_i} \prod_{i \in J^c} \omega_i^{s_i+e-x_i} = \prod_{i \in T} \omega_i^{t'_i} \prod_{i \in J'} \omega_i^{e-1-x'_i} \prod_{i \in J'^c} \omega_i^{s'_i+e-x'_i} \tag{4.10.5}$$

Then a basis for the intersection of $L_{V_{\vec{t},\vec{s}}}(\chi_1, \chi_2)$ with $L_{V_{\vec{t'},\vec{s'}}}(\chi_1, \chi_2)$ is given by c_α for $\alpha \in J^{AH}_{V_{\vec{t},\vec{s}}}(\chi_1, \chi_2) \cap J^{AH}_{V_{\vec{t'},\vec{s'}}}(\chi_1, \chi_2)$, together with c_{un} and/or c_{tr} if $\chi_2^{-1}\chi_1$ is trivial and/or cyclotomic with some additional conditions.

When $e = 1$, there is another algorithm to specify a basis of $L_{V_{\vec{t},\vec{s}}}(\chi_1, \chi_2)$, given in [DDR]. We recall some essentials of this algorithm because it will be convenient/shorter to use it for some of the calculations in the unramified case.

Definition 4.11. Let $J_{max} := \{i \in \mathbb{Z}/f\mathbb{Z} | y_i = 0\}$, where y_i are as defined in Definition 4.3.

Definition 4.12. Let $(a_{f-1}, ..., a_0)$ be the tame signature of $\chi_2^{-1}\chi_1$. The function $\delta : \mathbb{Z} \to \mathbb{Z}$ is defined in the following way: For $j \in \mathbb{Z}$, $\delta(j) = j$ unless $(a_{i-1}, a_{i-2}, ..., a_j) = (p, p-1, ..., p-1)$ for some $j < i$, in which case $\delta(j) = i$. When $j = i-1$, the condition $(a_{i-1}, a_{i-2}, ..., a_j) = (p, p-1, ..., p-1)$ is interpreted as $a_j = p$.

δ induces a function $\mathbb{Z}/f\mathbb{Z} \to \mathbb{Z}/f\mathbb{Z}$, also denoted by δ.

Let J be a subset of $\mathbb{Z}/f\mathbb{Z}$. If $\delta(J) \subset J$, $\mu(J) := J$. Else choose some $[i_1] \in \delta(J) \setminus J$ and let j_1 be the largest integer such that $j_1 < i_1$, $[j_1] \in J$ and $\delta(j_1) = i_1$. If $J = \{[j_1], ..., [j_r]\}$ with $j_1 > j_2 > ... > j_r > j_1 - f$, define i_κ for $\kappa \in [2, r]$ inductively as follows:

$$i_\kappa = \begin{cases} \delta(j_\kappa), & \text{if } i_{\kappa-1} > \delta(j_\kappa) \\ j_\kappa, & \text{otherwise} \end{cases}$$

Then $\mu(J) := \{[i_1], ..., [i_r]\}$.

When $e = 1$, $L_{V_{\vec{t},\vec{s}}}(\chi_1, \chi_2)$ has a basis given by certain elements of $\mathrm{Ext}^1_{\mathrm{G}_K}(\overline{\mathbb{F}}(\chi_2), \overline{\mathbb{F}}(\chi_1))$ indexed by $\tau \in \mu(J_{max})$ along with c_{un} and/or c_{tr} if $\chi_2^{-1}\chi_1$ is trivial and/or cyclotomic with some additional conditions.

We now state the criterion for determining the Serre weights associated to an irreducible G_K representation, when $K = \mathbb{Q}_p$. It can be stated for arbitrary K, but this paper will only need the case $K = \mathbb{Q}_p$.

Let $\overline{\rho}$ be an irreducible $G_{\mathbb{Q}_p}$ representation. Let η_1 and η_2 be the two level 2 fundamental characters of $I_{\mathbb{Q}_p}$. $V_{t,s}^\vee$ is a Serre weight of $\overline{\rho}^\vee$ iff $\overline{\rho} = \eta_1^{t+s}\eta_2^t \oplus \eta_1^t\eta_2^{t+s}$.

4.13. **Translation from Linear Algebra to Geometry of Stacks.** Let $V_{\vec{t},\vec{s}}$ and $V_{\vec{t'},\vec{s'}}$ be non-isomorphic, non-Steinberg Serre weights. The closure of irreducible $\overline{\mathbb{F}}$-representations is 0-dimensional. Thus, unless $K = \mathbb{Q}_p$, we only need to consider closures of families of reducible representations in order to detect codimension 1 intersections between irreducible components. On the other hand, if $K = \mathbb{Q}_p$, then by Proposition 3.4, we additionally need to consider when $\mathcal{X}_{V_{t,s}} \cap \mathcal{X}_{V_{t',s'}}$ contains irreducible finite type points. This last point is dealt with easily.

Lemma 4.14. *When $K = \mathbb{Q}_p$, $\mathcal{X}_{V_{t,s}} \cap \mathcal{X}_{V_{t',s'}}$ contains irreducible finite type points if and only if $s' = p-1-s$ and $t' \equiv t+s \mod p-1$.*

Proof. By the algorithm for computing Serre weights, we need to determine when

$$\eta_1^{t+s}\eta_2^t \oplus \eta_1^t\eta_2^{t+s} = \eta_1^{t'+s'}\eta_2^{t'} \oplus \eta_1^{t'}\eta_2^{t'+s'}.$$

Since $V_{t,s}$ and $V_{t',s'}$ are non-isomorphic and non-Steinberg, the relationship between (t,s) and (t',s') follows immediately. □

Remark 4.15. The criterion in the statement of Lemma 4.14 is the same as that in Proposition 2.1(ii)(b).

Using Proposition 3.4, we can state a sufficient (and necessary when $K \neq \mathbb{Q}_p$) condition for $\mathcal{X}_{V_{\vec{t},\vec{s}}} \cap \mathcal{X}_{V_{\vec{t'},\vec{s'}}}$ to be codimension 1: there exist G_K characters χ_1 and χ_2 so that after replacing χ_1 and χ_2 by generic unramified twists, the subspace

$\{\overline{\rho} \mid V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}} \in W(\overline{\rho})\} \subset \mathrm{Ext}^1_{\mathrm{G}_K}(\overline{\mathbb{F}}(\chi_1^{-1}), \overline{\mathbb{F}}(\chi_2^{-1}))$ has dimension $ef-1$. By generic unramified twists we mean that if we let $\mathbb{G}_m \times \mathbb{G}_m$ parametrize the unramified twists of χ_1 and χ_2 via the value of the unramified characters on Frob_K, then the statement is true for the points of a dense open subset of $\mathbb{G}_m \times \mathbb{G}_m$. Equivalently, $L_{V_{\vec{t},\vec{s}}}(\chi_1,\chi_2) \cap L_{V_{\vec{t'},\vec{s'}}}(\chi_1,\chi_2) \subset \mathrm{Ext}^1_{\mathrm{G}_K}(\overline{\mathbb{F}}(\chi_2), \overline{\mathbb{F}}(\chi_1))$ is spanned by $ef-1$ basis elements excluding c_{un} and c_{tr}.

Therefore, we must find G_K characters χ_1 and χ_2 such that there exist subsets J and J' of T, and for each $i \in T$, there exist $x_i, x_i' \in [0, e-1]$ such that (4.10.4) and (4.10.5) are satisfied. We next require that $|J^{AH}_{V_{\vec{t},\vec{s}}}(\chi_1,\chi_2) \cap J^{AH}_{V_{\vec{t'},\vec{s'}}}(\chi_1,\chi_2)| = ef-1$. This can happen in one of two ways.

Definition 4.16. We say that a pair of Serre weights $V_{\vec{t},\vec{s}}$ and $V_{\vec{t'},\vec{s'}}$ have a *type I* intersection witnessed by (χ_1,χ_2) if $|J^{AH}_{V_{\vec{t},\vec{s}}}(\chi_1,\chi_2)| = ef$ while $|J^{AH}_{V_{\vec{t'},\vec{s'}}}(\chi_1,\chi_2)| = ef-1$. The ordering of the pair of Serre weights is not important for this definition.

Definition 4.17. We say that a pair of Serre weights $V_{\vec{t},\vec{s}}$ and $V_{\vec{t'},\vec{s'}}$ have a *type II* intersection witnessed by (χ_1,χ_2) if $J^{AH}_{V_{\vec{t},\vec{s}}}(\chi_1,\chi_2) = J^{AH}_{V_{\vec{t'},\vec{s'}}}(\chi_1,\chi_2)$ of cardinality $ef-1$.

We will say that the number of separated families in a type I (resp. type II) intersection for the Serre weights $V_{\vec{t},\vec{s}}$ and $V_{\vec{t'},\vec{s'}}$ is n if there exist exactly n pairs of G_K characters that witness the type I (resp. type II) intersection, such that each pair is distinct from all others upon restriction to I_K. By Proposition 3.4, if $K \neq \mathbb{Q}_p$, the number of irreducible components of $\mathcal{X}_{V_{\vec{t},\vec{s}}} \cap \mathcal{X}_{V_{\vec{t'},\vec{s'}}}$ of dimension $[K:\mathbb{Q}_p]-1$ equals the number of separated families in either a type I or a type II intersection for the Serre weights $V_{\vec{t},\vec{s}}$ and $V_{\vec{t'},\vec{s'}}$.

For the remainder of this article, we may assume that $\vec{s}, \vec{s'}$ do not have all components equal to $p-1$ since we are excluding Steinberg components from analysis. Finally, since we are interested in intersections of different irreducible components, we may assume that $V_{\vec{t},\vec{s}} \neq V_{\vec{t'},\vec{s'}}$.

Lemma 4.18. $|J^{AH}_{V_{\vec{t},\vec{s}}}(\chi_1,\chi_2)| = ef$ *if and only if* $\chi_1 = \prod_{i\in T} \omega_i^{s_i+e} \prod_{i \in T} \omega_i^{t_i}$ *and* $\chi_2|_{I_K} = \prod_{i\in T}\omega_i^{t_i}$.

Proof. By Remarks 4.4 and 4.7, $|J^{AH}_{V_{\vec{t},\vec{s}}}(\chi_1,\chi_2)| = ef$ implies that $\chi_1|_{I_K} = \prod_{i\in T}\omega_i^{z_i} = \prod_{i\in T}\omega_i^{s_i+e}$ and $\chi_2|_{I_K} = \prod_{i \in T}\omega_i^{y_i} = 1$. On the other hand, starting with these χ_1 and χ_2, we can compute $A_{\min}$ as in Definition 4.2. In this case, we observe that $(m_{f-1}, ..., m_0) = (0, ..., 0)$ as $\chi_2|_{I_K} = 1$. $J = \emptyset$ satisfies the criterion for $A_{\min}$, and we obtain that $y_i = 0$ and $z_i = s_i + e$ for all i. By Remark 4.7, we get that $|J^{AH}_{V_{\vec{0},\vec{s}}}(\chi_1,\chi_2)| = ef$, as desired. □

Definition 4.19. Let χ_1, χ_2 be two characters and $V_{\vec{t},\vec{s}}$ be a Serre weight satisfying the conditions in Lemma 4.18. Then, we say that $V_{\vec{t},\vec{s}}$ is the *highest weight* associated to the pair (χ_1,χ_2). It is uniquely determined since not all s_i can be $p-1$. Moreover, knowing the highest weight determines the pair (χ_1,χ_2) after restriction to I_K.

Let $V^{\vee}_{\vec{t},\vec{s}} = V_{\vec{t},\vec{s}}$. Then for any $\overline{\rho}^{\vee} \in \mathrm{Ext}^1_{\mathrm{G}_K}(\overline{\mathbb{F}}(\chi_2), \overline{\mathbb{F}}(\chi_1))$, we say that $\mathcal{X}_{V_{\vec{t},\vec{s}}}$ is the *highest weight component* containing $\overline{\rho}$.

Remark 4.20. The number of separated families in a type I intersection can be at most 2, because one of the two Serre weights has to be the highest weight.

Lemma 4.21. *$|J^{AH}_{V_{\vec{t'},\vec{s'}}}(\chi_1,\chi_2)| = ef-1$ if and only if one of the following conditions is satisfied:*

(i) There exists an $i \in T$ such that $\chi_1 = \omega_i^{e-1}\prod_{j\neq i}\omega_j^{s_j+e}\prod_{j\in T}\omega_j^{t_j}$ and $\chi_2 = \omega_i^{s_i+1}\prod_{j\in T}\omega_j^{t_j}$, and moreover, $s_i \leq p-2$, and if $f=1$ then $s_i < p-2$.

(ii) $e=1, f>1$ and there exists an $i \in T$ such that $\chi_1 = \omega_i^{e-1}\prod_{j\neq i}\omega_j^{s_j+e}\prod_{j\in T}\omega_j^{t_j}$, $\chi_2 = \omega_i^{s_i+1}\prod_{j\in T}\omega_j^{t_j}$, $s_i = p-1$ and $s_{i-1} > 0$.

(iii) $e>1$ and there exists an $i\in T$ such that $\chi_1 = \omega_i^{s_i+e-1}\prod_{j\neq i}\omega_j^{s_j+e}\prod_{j\in T}\omega_j^{t_j}$, $\chi_2 = \omega_i\prod_{j\in T}\omega_j^{t_j}$, and $s_i \neq 0$.

In the first two situations above, $y_i = s_i+1$ and $y_j = 0$ for all $j\neq i$ (recall Definition 4.3). On the other hand, if $y_i = s_i+1$ and $y_j=0$ for all $j \neq i$, then one of the two above must be satisfied.

The third situation is equivalent to $y_i = 1$, $y_j = 0$ for all $j\neq i$ along with $s_i \neq 0$.

Proof. By Remark 4.7, if $|J^{AH}_{V_{\vec{t'},\vec{s'}}}(\chi_1,\chi_2)| = ef-1$ then one of the following two conditions must be satisfied:

(i) There exists $i\in T$ such that $y_i = s_i+1$ and for $j\neq i$, $y_j = 0$. This implies that $\chi_1 = \omega_i^{e-1}\prod_{j\neq i}\omega_j^{s_j+e}\prod_{j\in T}\omega_j^{t_j}$ and $\chi_2 = \omega_i^{s_i+1}\prod_{j\in T}\omega_j^{t_j}$. On the other hand, starting with such χ_1 and χ_2, twisting them by $\prod_{j\in T}\omega_j^{-t_j}$ and applying the recipe to compute $A_{\min}$ (Definition 4.10), y_j and z_j (Definition 4.3), we branch into two scenarios:
   (a) If $s_i \leq p-2$ for $f>1$ and $< p-2$ for $f=1$, then $A_{\min} = \emptyset$, $y_i = s_i+1$ and $y_j = 0$ for $j\neq i$, giving $|J^{AH}_{V_{\vec{t},\vec{s}}}(\chi_1,\chi_2)| = ef-1$.
   (b) If $s_i = p-1$, then $\chi_2 \otimes \prod_{j\in T}\omega_j^{-t_j} = \prod_{j\in T}\omega_j^{m_j}$, where $m_{i-1} = 1$ and $m_j = 0$ if $j \neq i-1$. Note that this automatically implies that $f>1$, since we are assuming our Serre weights are non-Steinberg. We can obtain the desired values of y_j's if and only if $(m_{f-1},...,m_0)$ is not already in $\mathcal{S}$ of Definition 4.2. In other words, if and only if $e=1$ and $s_{i-1}\neq 0$.

(ii) There exists $i\in T$ such that $y_i = 1$, $s_i \neq 0$ (this is to enforce distinction from the condition above) and $y_j = 0$ when $j\neq i$. Note that this automatically implies that $e>1$, and that $\chi_1 = \omega_i^{s_i+e-1}\prod_{j\neq i}\omega_j^{s_j+e}\prod_{j\in T}\omega_j^{t_j}$ and $\chi_2 = \omega_i\prod_{j\in T}\omega_j^{t_j}$. On the other hand, starting with such χ_1 and χ_2, twisting them by $\prod_{j\in T}\omega_j^{-t_j}$ and applying the recipe to compute $A_{\min}$, y_j and z_j, we obtain that $A_{\min} = \emptyset$, $y_i = 1$ and $y_j = 0$ for $i\neq j$, and we get the correct cardinality of $J^{AH}_{V_{\vec{t},\vec{s}}}(\chi_1,\chi_2)$.

□

Remark 4.22. In the cases Lemma 4.21(i) and Lemma 4.21(ii),

$$\chi_2^{-1}\chi_1 = \omega_i^{e-2-s_i}\prod_{j\neq i}\omega_j^{s_j+e}$$

In the case Lemma 4.21(iii),

$$\chi_2^{-1}\chi_1 = \omega_i^{e-2+s_i}\prod_{j\neq i}\omega_j^{s_j+e}$$

Remark 4.23. When $e = 1$, the cases Lemma 4.21(i) and Lemma 4.21(ii) are together equivalent to $J_{\max} = \mathbb{Z}/f\mathbb{Z} \smallsetminus [i]$.

Before launching into computations of Type I and II intersections, we introduce some more notation. When comparing f-tuples $\vec{s}$ and $\vec{s'}$, we will often only state the values of s_i and s'_i that have specific constraints or are potentially different from each other. If the values of s_i or s'_i are not specified, then we assume that $s_i = s'_i$. If no range is specified for s_i, we mean that beyond any relations that it must satisfy with respect to s'_i, the value of s_i can be anything in $[0, p-1]$. Further, if we say $(..., s_i, ...) = (..., \in [a, b], ...)$, we mean that s_i can take any value $\in [a, b]$. Similar notational assumptions apply with the roles of s_i and s'_i interchanged. Finally, we say that a tuple $(b_{f-1}, b_{f-2}, ..., b_0)$ is equivalent to $(b'_{f-1}, b'_{f-2}, ..., b'_0)$ if $\sum_{j=0}^{f-1} b_j p^{f-1-j} \equiv \sum_{j=0}^{f-1} b'_j p^{f-1-j} \mod p^f - 1$.

We will retain the symbols y_i, z_i, $\mathcal{I}_i$, λ_i and ξ_i as defined in Definitions 4.3, 4.5 and 4.9 for $V_{\vec{t},\vec{s}}$, and will replace them respectively with y'_i, z'_i, $\mathcal{I}'_i$, λ'_i and ξ'_i for $V_{\vec{t'},\vec{s'}}$ and with y''_i, z''_i, $\mathcal{I}''_i$, λ''_i and ξ''_i for $V_{\vec{t''},\vec{s''}}$.

5. Type I intersections

In this section, we will compute criteria for existence of a pair of characters (χ_1, χ_2) witnessing a type I intersection for Serre weights $V_{\vec{t},\vec{s}}$ and $V_{\vec{t'},\vec{s'}}$, with $|J^{AH}_{V_{\vec{t},\vec{s}}}(\chi_1,\chi_2)| = ef$ and $|J^{AH}_{V_{\vec{t'},\vec{s'}}}(\chi_1,\chi_2)| = ef - 1$. $|J^{AH}_{V_{\vec{t'},\vec{s'}}}(\chi_1,\chi_2)| = ef - 1$ can happen via one of three ways as enumerated in Lemma 4.21. In all three situations, we may assume without loss of generality that i in the statements of Lemma 4.21(i), Lemma 4.21(ii) and Lemma 4.21(iii) is $f-1$. We will also count the number of families contributing to a type I intersection when $V_{\vec{t},\vec{s}}$ and $V_{\vec{t'},\vec{s'}}$ are both weakly regular. (In the general case, the information can still be gleaned directly from the computations that follow, but we omit the explicit description for the sake of clarity).

5.1. **Type I intersections when $f = 1$.** We will omit subscripts of components of f-tuples in this section as $f = 1$.

5.1.1. *Case 1.* : $|J^{AH}_{V_{\vec{t'},\vec{s'}}}(\chi_1,\chi_2)| = ef - 1$ via Lemma 4.21(i).

Suppose $p = 2$. The non-Steinberg condition requires that $s = s' = 0$. Plugging in s and s' in the expressions for χ_2 (using Lemmas 4.18 and 4.21), we get $t \equiv t' + 1 \mod p-1$. This gives $t = t'$ and shows that $V_{\vec{t},\vec{s}}$ and $V_{\vec{t'},\vec{s'}}$ are isomorphic, a contradiction. Therefore, we may assume $p > 2$.

Comparing the two ways of writing $\chi_2^{-1}\chi_1$, we obtain:

$$s + e \equiv e - 2 - s' \mod p-1 \iff$$
$$s' \equiv -2 - s \equiv p - 3 - s \mod p-1$$

This gives one of the following two situations:

(1) $s \le p-3 \implies s' = p - s - 3$.

(2) $s = p-2 \implies s' = p-2$.

In both situations, comparing the two ways of writing χ_2, we obtain that $t' + s' + 1 \equiv t \mod p-1$. In other words $t' \equiv t + p - s' - 2 \equiv t + s + 1 \mod p-1$.

The second situation therefore implies that $V_{\vec{t},\vec{s}} = V_{\vec{t'},\vec{s'}}$, which is a contradiction. The first situation is equivalent to the conditions in Proposition 2.1(ii)(a) implying that $\mathrm{Ext}^1_{\mathbb{F}[\mathrm{GL}_2(k)]}(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}}) \neq 0$. Since it is symmetric in s and s', whenever $\mathrm{Ext}^1_{\mathbb{F}[\mathrm{GL}_2(k)]}(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}}) \neq 0$, there exist two separated families witnessing a type I intersection for $V_{\vec{t},\vec{s}}$ and $V_{\vec{t'},\vec{s'}}$.

5.1.2. *Case 2.* : $|J^{AH}_{V_{\vec{t'},\vec{s'}}}(\chi_1, \chi_2)| = ef - 1$ via Lemma 4.21(iii). Implicit in this case is $e > 1$ and $p > 2$, the latter since s' is not allowed to be 0.

Comparing the two ways of writing $\chi_2^{-1}\chi_1$, we obtain:

$$\begin{aligned} s + e \equiv e - 2 + s' \mod p-1 \iff \\ s' \equiv s + 2 \mod p-1 \end{aligned} \tag{5.1.1}$$

This gives one of the following two situations:

(1) $s < p-3 \implies s' = s+2$.
(2) $s = p-3 \implies s' = 0$.
(3) $s = p-2 \implies s' = 1$.

In both situations, comparing the two ways of writing χ_2, we obtain that $t' \equiv -1 + t \equiv p - 2 + t \mod p-1$. By comparing with Proposition 2.1, we notice that the first situation implies $\mathrm{Hom}_{\mathrm{GL}_2(k)}(V_{\vec{t},\vec{s}}, H^1(\mathrm{G}_K, V_{\vec{t'},\vec{s'}})) \neq 0$ (Proposition 2.6), the second implies $\mathrm{Ext}^1_{\mathbb{F}[\mathrm{GL}_2(k)]}(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}}) \neq 0$ via Proposition 2.1(ii)(a) and the third implies $\mathrm{Ext}^1_{\mathbb{F}[\mathrm{GL}_2(k)]}(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}}) \neq 0$ via Proposition 2.1(ii)(b). Notice that the relationship between s and s' is asymmetric in all three situations, unless $p = 3$ in which case the second and third situations are symmetric.

The above calculations may be summarized in the following proposition:

Proposition 5.2. *Let $f = 1$. A Type I intersection occurs with non-isomorphic, non-Steinberg Serre weights $V_{\vec{t},\vec{s}}$ and $V_{\vec{t'},\vec{s'}}$ if and only if one of the following holds true:*

(i) $\mathrm{Ext}^1_{\mathbb{F}[\mathrm{GL}_2(k)]}(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}}) \neq 0$ *via Proposition 2.1(ii)(a). In this case, 2 families witness the type I intersection.*
(ii) $e > 1$ *and* $\mathrm{Hom}_{\mathrm{GL}_2(k)}(V_{\vec{t},\vec{s}}, H^1(\mathrm{G}_K, V_{\vec{t'},\vec{s'}})) \neq 0$. *In this case, 1 family witnesses the type I intersection.*
(iii) $e > 1$ *and* $s = p-2$, $s' = 1$, $d' \equiv -1 + d \mod p-1$. *In this case, the number of families witnessing the type I intersection is 1 unless $p = 3$, in which case the number is 2.*

Note that the non-isomorphic, non-Steinberg condition automatically forces $p > 2$. Further, the last statement implies $\mathrm{Ext}^1_{\mathbb{F}[\mathrm{GL}_2(k)]}(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}}) \neq 0$ *via Proposition 2.1(ii)(b).*

5.3. **Type I intersections when $f > 1$.**

5.3.1. *Case 1.* : $|J^{AH}_{V_{\vec{t'},\vec{s'}}}(\chi_1,\chi_2)| = ef-1$ via Lemma 4.21(i) or via Lemma 4.21(ii).

Comparing the two ways of writing $\chi_2^{-1}\chi_1$, we obtain the following equivalences mod p^f-1 upto translating all indices by a fixed element of $\mathbb{Z}/f\mathbb{Z}$:

$$\begin{aligned}
\sum_{j\in T} p^{f-1-j}(s_j+e) &\equiv e-2-s'_{f-1}+\sum_{j=0}^{f-2} p^{f-1-j}(s'_j+e) \qquad \Longleftrightarrow \\
\sum_{j\in T} p^{f-1-j}s_j &\equiv -2-s'_{f-1}+\sum_{j=0}^{f-2} p^{f-1-j}s'_j \\
&\equiv p-s'_{f-1}-2+p(s'_{f-2}-1)+\sum_{j=0}^{f-3} p^{f-1-j}s'_j
\end{aligned}$$

Therefore, for a fixed $\vec{s}$, $\vec{s'}$ is forced to be unique since each $s'_i \in [0,p-1]$, and the non-Steinberg condition requires that not all s'_i can be $p-1$. Similarly, for a fixed $\vec{s'}$, $\vec{s}$ is forced to be unique.

We have a number of possible cases :

(i) Suppose $s'_{f-1} \le p-2$, $s'_{f-2} = s'_{f-3} = ... = s'_{f-i} = 0$ and $s'_{f-1-i} \ge 1$, where $i \ge 1$.

$$\begin{aligned}
(p-s'_{f-1}-2, s'_{f-2}-1, s'_{f-3}, ..., s'_{f-i}, s'_{f-1-i}, s'_{f-2-i}, ..., s'_0) &\equiv \\
(p-s'_{f-1}-2, -1, 0, ..., 0, s'_{f-1-i}, s'_{f-2-i}, ..., s'_0) &\equiv \\
(p-s'_{f-1}-2, p-1, p-1, ..., p-1, s'_{f-1-i}-1, s'_{f-2-i}, ..., s'_0)&
\end{aligned}$$

Therefore, $s_{f-1} = p-s'_{f-1}-2$, $s_{f-2} = s_{f-3} = ... = s_{f-i} = p-1$, $s_{f-1-i} = s'_{f-1-i}-1 \le p-2$ and $s_j = s'_j$ for all the remaining j's.

(ii) Suppose $s'_{f-1} \le p-3$, $s'_{f-2} = s'_{f-3} = ... = s'_0 = 0$.

$$\begin{aligned}
(p-s'_{f-1}-2, s'_{f-2}-1, s'_{f-3}, ..., s'_0) &\equiv (p-s'_{f-1}-2, -1, 0, ..., 0) \\
&\equiv (p-s'_{f-1}-3, p-1, p-1, ..., p-1)
\end{aligned}$$

We get $s_{f-1} = p-s'_{f-1}-3 \le p-3$, $s_{f-2} = s_{f-3} = ... = s_0 = p-1$.

(iii) Suppose $s'_{f-1} = p-2$, $s'_{f-2} = s'_{f-3} = ... = s'_0 = 0$.

$$\begin{aligned}
(p-s'_{f-1}-2, s'_{f-2}-1, s'_{f-3}, ..., s'_0) &\equiv (0, -1, 0, ..., 0) \\
&\equiv (p-1, p-2, p-1, ..., p-1)
\end{aligned}$$

Hence, $s_{f-2} = p-2$ and all the other s_j's equal $p-1$.

The remaining cases require $|J^{AH}_{V_{\vec{t'},\vec{s'}}}(\chi_1,\chi_2)| = ef-1$ via Lemma 4.21(ii), and implicitly, $e=1$.

(iv) Suppose $s'_{f-1} = p-1$, $s'_{f-2} > 1$.

$$\begin{aligned}
(p-s'_{f-1}-2, s'_{f-2}-1, s'_{f-3}, ..., s'_0) &\equiv (-1, s'_{f-2}-1, s'_{f-3}, ..., s'_0) \\
&\equiv (p-1, s'_{f-2}-2, s'_{f-3}, ..., s'_0)
\end{aligned}$$

Therefore, $s_{f-1} = p-1$, $s_{f-2} = s'_{f-2}-2$ and $s_j = s'_j$ for the remaining j's.

(v) Suppose $f > 2$, $s'_{f-1} = p-1$, $s'_{f-2} = 1$, $s'_{f-3} = s'_{f-4} = ... = s'_{f-i} = 0$ and $s'_{f-1-i} \geq 1$ for some $i > 2$.

$$\begin{aligned}(p - s'_{f-1} - 2, s'_{f-2} - 1, s'_{f-3}, ..., s'_{f-i}, s'_{f-1-i}, s'_{f-2-i}, ..., s'_0) \equiv \\ (-1, 0, 0, ..., 0, s'_{f-1-i}, s'_{f-2-i}, ..., s'_0) \equiv \\ (p-1, p-1, p-1, ..., p-1, s'_{f-1-i} - 1, s'_{f-2-i}, ..., s'_0)\end{aligned}$$

Therefore, $s_{f-1} = s_{f-2} = ... = s_{f-i} = p-1$, $s_{f-1-i} = s'_{f-1-i} - 1$ and $s_j = s'_j$ for all the other j's.

(vi) Suppose $f = 2$, $s'_{f-1} = p-1$, $s'_{f-2} = 1$.

$$(p - s'_{f-1} - 2, s'_{f-2} - 1) \equiv (-1, 0) \equiv (p-2, p-1)$$

Therefore, $s_{f-1} = p-2$ and $s_{f-2} = p-1$.

(vii) Suppose $f > 2$, $s'_{f-1} = p-1$, $s'_{f-2} = 1$ and $s'_{f-3} = ... = s'_0 = 0$.

$$\begin{aligned}(p - s'_{f-1} - 2, s'_{f-2} - 1, s'_{f-3}, ..., s'_0) &\equiv (-1, 0, 0, ..., 0) \\ &\equiv (p-2, p-1, p-1, ..., p-1)\end{aligned}$$

Therefore, $s_{f-1} = p-2$ and $s_j = p-1$ for all the other j's.

The results of the computations are summarized in the proposition below.

Proposition 5.4. *Let $f > 1$. Consider pairs $(\vec{s}, \vec{s'})$ satisfying:*

- $\prod_{j=0}^{f-1} \omega_j^{s_j} = \omega_{f-1}^{-s'_{f-1}-2} \prod_{j=0}^{f-2} \omega_j^{s'_j}$*, where $s_j, s'_j \in [0, p-1]$;*
- *Not all s_j, as well as not all s'_j, are $p-1$.*
- *$y'_{f-1} = s'_{f-1} + 1$ and $y'_j = 0$ for $j \neq f-1$ (y'_j are as defined in Definition 4.3).*

Below is an enumeration of all such pairs.

(i) $(s_{f-1}, s_{f-2}, ..., s_{f-i}, s_{f-1-i}) = (\in [0, p-2], p-1, ..., p-1, \in [0, p-2])$*, where $i \in [1, f-1]$;*
$(s'_{f-1}, s'_{f-2}, ..., s'_{f-i}, s'_{f-1-i}) = (p - s_{f-1} - 2, 0, ..., 0, s_{f-1-i} + 1)$.

(ii) $(s_{f-1}, s_{f-2}, ..., s_0) = (\in [0, p-3], p-1, ..., p-1)$*;*
$(s'_{f-1}, s'_{f-2}, ..., s'_0) = (p - 3 - s_{f-1}, 0, ..., 0)$.
This only makes sense if $p \geq 3$.

(iii) $(s_{f-1}, s_{f-2}, s_{f-3}, ..., s_0) = (p-1, p-2, p-1, ..., p-1)$*;*
$(s'_{f-1}, s'_{f-2}, s'_{f-3}, ..., s'_0) = (p-2, 0, 0, ..., 0)$.

When $e = 1$, we additionally have:

(iv) $(s_{f-1}, s_{f-2}) = (p-1, \in [0, p-3])$*;*
$(s'_{f-1}, s'_{f-2}) = (p-1, s_{f-2} + 2)$.
This only makes sense if $p \geq 3$.

(v) $f > 2$*,*
$(s_{f-1}, s_{f-2}, s_{f-3}, ..., s_{f-i}, s_{f-1-i}) = (p-1, p-1, p-1, ..., p-1, \in [0, p-2])$*, where $i \in [2, f-1]$;*

$(s'_{f-1}, s'_{f-2}, s'_{f-3}..., s'_{f-i}, s'_{f-1-i}) = (p-1, 1, 0, ..., 0, s_{f-1-i}+1)$.

(vi) $f = 2$,
$(s_{f-1}, s_{f-2}) = (p-2, p-1)$;
$(s'_{f-1}, s'_{f-2}) = (p-1, 1)$.

(vii) $f > 2$,
$(s_{f-1}, s_{f-2}, s_{f-3}, ..., s_0) = (p-2, p-1, p-1, ..., p-1)$;
$(s'_{f-1}, s'_{f-2}, s'_{f-3}, ..., s'_0) = (p-1, 1, 0, ..., 0)$.

Comparing the two ways of writing χ_2, we obtain:

$$\sum_{j\in T} p^{f-1-j} t'_j + (s'_{f-1}+1) \equiv \sum_{j\in T} p^{f-1-j} t_j \mod p^f - 1 \iff$$

$$\sum_{j\in T} p^{f-1-j} t'_j \equiv -1 - s'_{f-1} + \sum_{j\in T} p^{f-1-j} t_j \mod p^f - 1 \tag{5.4.1}$$

5.4.1. *Case 2.* : $|J^{AH}_{V_{\vec{t'},\vec{s'}}}(\chi_1,\chi_2)| = ef-1$ via Lemma 4.21(iii). Implicit in this case is that $e > 1$.

Comparing the two ways of writing $\chi_2^{-1}\chi_1$, we obtain the following equivalences mod $p^f - 1$:

$$\begin{aligned}
\sum_{j\in T} p^{f-1-j}(s_j+e) &\equiv e - 2 + s'_{f-1} + \sum_{j\neq f-1} p^{f-1-j}(s'_j+e) \iff \\
\sum_{j\in T} p^{f-1-j}(s_j+1) &\equiv s'_{f-1} - 1 + \sum_{j\neq f-1} p^{f-1-j}(s'_j+1) \iff \\
\sum_{j\in T} p^{f-1-j} s_j &\equiv s'_{f-1} - 2 + \sum_{j\neq f-1} p^{f-1-j} s'_j
\end{aligned} \tag{5.4.2}$$

Proposition 5.5. *Let $f > 1$ and $e > 1$.*

Consider pairs $(\vec{s}, \vec{s'})$ satisfying:

- $\prod_{j=0}^{f-1} \omega_j^{s_j} = \omega_{f-1}^{s'_{f-1}-2} \prod_{j=0}^{f-2} \omega_j^{s'_j}$*, where $s_j, s'_j \in [0, p-1]$;*
- *Not all s_j, as well as not all s'_j, are $p-1$. Also, $s'_{f-1} \neq 0$*
- *$y'_{f-1} = 1$ and $y'_j = 0$ for $j \neq f-1$ (y'_j are as defined in Definition 4.3).*
- *After reindexing if necessary, $\vec{s}$ and $\vec{s'}$ satisfy one of the below:*
  - *(i)* $s_{f-1} \leq p-3$;
    $s'_{f-1} = s_{f-1}+2$.
    This only makes sense if $p \geq 3$.
  - *(ii)* $(s_{f-1}, s_{f-2}, ..., s_{f-i}, s_{f-1-i}) = (p-1, p-1, ..., p-1, \in [0, p-2])$*, where $i \geq 1$;*
    $(s'_{f-1}, s'_{f-2}, ..., s'_{f-i}, s'_{f-1-i}) = (1, 0, ..., 0, s_{f-i-1}+1)$.
  - *(iii)* $(s_{f-1}, s_{f-2}, ..., s_0) = (p-2, p-1, ..., p-1)$;
    $(s'_{f-1}, s'_{f-2}, ..., s'_0) = (1, 0, ..., 0)$.

Comparing the two ways of writing χ_2, we obtain:

$$\sum_{j\in T} p^{f-1-j}t'_j \equiv -1 + \sum_{j\in T} p^{f-1-j}t_j \mod p^f - 1 \tag{5.5.1}$$

It is evident that when each s_i and each s'_i is $< p-1$, the relationship between $\vec{s}$ and $\vec{s'}$ described in Propositions 5.4 and 5.5 is asymmetric.

The calculations for type 1 intersections are summarized below.

Proposition 5.6. *Let $f > 1$. $V_{\vec{t},\vec{s}}$ and $V_{\vec{t'},\vec{s'}}$ be a pair of non-isomorphic, non-Steinberg Serre weights. Then there exist G_K characters χ_1 and χ_2 such that $|J^{AH}_{V_{\vec{t},\vec{s}}}(\chi_1,\chi_2)| = ef$, $|J^{AH}_{V_{\vec{t'},\vec{s'}}}(\chi_1,\chi_2)| = ef-1$ if and only if one of the following is satisfied:*

- *(i) Upto translating the indices by any fixed number, $\vec{s}$ and $\vec{s'}$ satisfy one of the conditions in Proposition 5.4 while $\vec{t}$ and $\vec{t'}$ satisfy (5.4.1) .*
- *(ii) Upto translating the indices by any fixed number, $\vec{s}$ and $\vec{s'}$ satisfy one of the conditions in Proposition 5.5 while $\vec{t}$ and $\vec{t'}$ satisfy (5.5.1).*

Corollary 5.7. Suppose $f > 1$ and $V_{\vec{t},\vec{s}}$ and $V_{\vec{t'},\vec{s'}}$ are two non-isomorphic weakly regular Serre weights. Then there exists a type I intersection for the pair if and only if one of the following holds (upto translating the indices by any fixed number and/or interchanging $V_{\vec{t},\vec{s}}$ and $V_{\vec{t'},\vec{s'}}$ if necessary):

- (i) $\vec{s}$ and $\vec{s'}$ satisfy Proposition 5.4(i) with $i = 1$; while $\vec{t}$ and $\vec{t'}$ satisfy (5.4.1).
- (ii) $\vec{s}$ and $\vec{s'}$ satisfy Proposition 5.5(i); while $\vec{t}$ and $\vec{t'}$ satisfy (5.5.1).

In other words, if and only if one of the following is true:

- (i) $\mathrm{Ext}^1_{\mathbb{F}[\mathrm{GL}_2(k)]}(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}}) \neq 0$, or
- (ii) $e > 1$ and $\mathrm{Hom}_{\mathrm{GL}_2(k)}(V_{\vec{t},\vec{s}}, H^1(\mathrm{G}_K, V_{\vec{t'},\vec{s'}})) \neq 0$

Equivalently, if and only if $\mathrm{Ext}^1_{\mathbb{F}[\mathrm{GL}_2(\mathcal{O}_K)]}(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}}) \neq 0$.

Moreover, exactly 1 family witnesses the type I intersection. When $f = 2$, $s_1 = \frac{p-1}{2}$, $s_0 = \frac{p-3}{2}$, $s'_1 = \frac{p-3}{2}$ and $s'_0 = \frac{p-1}{2}$, interchanging $\vec{s}$ and $\vec{s'}$ also satisfies Proposition 5.4(i) after shifting the indices by 1. However, in this case the computations of $\vec{t}$ and $\vec{t'}$ show that the situation is not symmetric, and we still have just 1 family witnessing the intersection.

Proof. By Propositions 2.1, 2.6 and 2.14 and corollary 2.10. □

6. Type II intersections

In this section, we will compute criteria for existence of a pair of characters (χ_1, χ_2) witnessing a type II intersection for (non-isomorphic and non-Steinberg) Serre weights $V_{\vec{t'},\vec{s'}}$ and $V_{\vec{t''},\vec{s''}}$. Thus, we will determine if χ_1 and χ_2 exist such that $J^{AH}_{V_{\vec{t'},\vec{s'}}}(\chi_1,\chi_2) = J^{AH}_{V_{\vec{t''},\vec{s''}}}(\chi_1,\chi_2)$ of cardinality $ef-1$. We will denote the highest weight associated to the pair by $V_{\vec{t},\vec{s}}$. A family witnessing a type II intersection necessarily also witnesses two type I intersections, one for the Serre weights $V_{\vec{t},\vec{s}}$ and $V_{\vec{t'},\vec{s'}}$, and the other for $V_{\vec{t},\vec{s}}$ and $V_{\vec{t''},\vec{s''}}$, thus it gives a triple intersection of codimension 1. On the other hand, every such triple intersection must involve a type II intersection.

6.1. **Type II intersections when $f = 1$.** We will omit subscripts of components of f-tuples in this section as $f = 1$.

6.1.1. *Case 1.* : $|J^{AH}_{V_{t',s'}}(\chi_1,\chi_2)| = |J^{AH}_{V_{t'',s''}}(\chi_1,\chi_2)| = ef-1$, both via Lemma 4.21(i) or both via Lemma 4.21(iii). It is immediate that this forces $V_{\vec{t'},\vec{s'}}$ and $V_{\vec{t''},\vec{s''}}$ to be isomorphic, a contradiction.

6.1.2. *Case 2.* : $|J^{AH}_{V_{t',s'}}(\chi_1,\chi_2)| = ef-1$ via Lemma 4.21(i) (Lemma 4.21(ii) is not possible because non-Steinberg); $|J^{AH}_{V_{t'',s''}}(\chi_1,\chi_2)| = ef-1$ via Lemma 4.21(iii). Lemma 4.21(iii) assumes that $e>1$.

Comparing ways of writing $\chi_2^{-1}\chi_1$ using Remark 4.22, we have

$$\begin{aligned} e-2-s' \equiv e-2+s'' \equiv s+e \mod p-1 \\ \iff s'' \equiv p-1-s', \\ s' \equiv p-3-s, \\ s'' \equiv s+2 \end{aligned} \tag{6.1.1}$$

Comparing ways of writing χ_2 using Lemma 4.21, we obtain:

$$\begin{aligned} s'+1+t' \equiv 1+t'' \equiv t \mod p-1 \\ \iff t'' \equiv t'+s' \\ t' \equiv t+s+1 \\ t'' \equiv -1+t \end{aligned} \tag{6.1.2}$$

By stipulation in Lemma 4.21(i), $s' \neq p-2$. Therefore, $s' \leq p-3$, and since $(t',s') \neq (t'',s'')$, $s<p-3$, the equivalences in (6.1.1) are equalities and $p>3$.

Notice the nature of type I intersection for $V_{\vec{t},\vec{s}}$ and $V_{\vec{t'},\vec{s'}}$. It corresponds to $\mathrm{Ext}^1_{\mathbb{F}[\mathrm{GL}_2(k)]}(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}}) \neq 0$ via Proposition 2.1(ii)(a). On the other hand, the type I intersection for $V_{\vec{t},\vec{s}}$ and $V_{\vec{t''},\vec{s''}}$ corresponds to $\mathrm{Hom}_{\mathrm{GL}_2(k)}(V_{\vec{t},\vec{s}}, H^1(\mathrm{G}_K, V_{\vec{t''},\vec{s''}})) \neq 0$.

Imposing the above conditions, we now calculate y', y'', z', z'', $\mathcal{I}'$, $\mathcal{I}''$, ξ' and ξ'', and compare $J^{AH}_{V_{t',s'}}(\chi_1,\chi_2)$ with $J^{AH}_{V_{t'',s''}}(\chi_1,\chi_2)$.

By Lemma 4.21, $y' = s'+1$ and $z' = e-1$, while $y''=1$ and $z''=s''+e-1$. Therefore

$$\begin{aligned} \mathcal{I}' &= [0,e-2] \\ \mathcal{I}'' &= \{1\} \cup [s''+1, s''+e-2] \\ \xi' &= (p-1)(e-1) + (e-2-s') \\ \xi'' &= (p-1)(s''+e-1) + (e-2+s'') \end{aligned}$$

As u' varies in $\mathcal{I}'$, $\xi' - u'(p-1) = (p-1)v' + (e-2-s')$ with v' taking up values in $[1,e-1]$. Similarly, as u'' varies in $\mathcal{I}''$,

$$\begin{aligned} \xi'' - u''(p-1) = & \quad (p-1)v'' + (e-2+s'') \quad \text{where } v'' \in [1,e-2] \cup \{s''+e-2\} \\ = & \quad (p-1)v'' + e-2-s' + (p-1) \quad \text{where } v'' \in [1,e-2] \cup \{p-3-s'+e\} \\ = & \quad (p-1)v'' + e-2-s', \quad \text{where } v'' \in [2,e-1] \cup \{p-2-s'+e\} \end{aligned}$$

By Definition 4.10, $J^{AH}_{V_{t',s'}}(\chi_1,\chi_2) = J^{AH}_{V_{t'',s''}}(\chi_1,\chi_2)$ if and only if for all $v' \in [1,e-1]$, there exists a $v'' \in [2,e-1] \cup \{p-2-s'+e\}$ such that:

$$\frac{(p-1)v'+(e-2-s')}{p^{\nu'}}=\frac{(p-1)v''+(e-2-s')}{p^{\nu''}} \tag{6.1.3}$$

where ν' is the p-adic valuation of the numerator on L.H.S, while ν'' is that of the numerator on R.H.S.

The only thing to check then is that (6.1.3) holds for $v'=1$ and $v''=p-2-s'+e$. Plugging in,

$$L.H.S.=\frac{p-3-s'+e}{p^{\nu}} \tag{6.1.4}$$

(6.1.5)
$$R.H.S.=\frac{(p-1)(p-2-s'+e)+e-2-s'}{p^{\nu''}}=\frac{p(p-3-s'+e)}{p^{\nu'+1}}=L.H.S.$$

Therefore, conditions (6.1.1) and (6.1.2) guarantee a type II intersection, and are equivalent to the conditions in Proposition 2.1(ii)(b). In this case, the relationship between the pairs (t',s') and (t'',s'') is symmetric except when $s'=1$ and $s''=p-2$. Therefore the calculations show the existence of 2 separated families (because the highest weights are distinct) witnessing the type II intersection except when $s'=1$ and $s''=p-2$. In the special case $s'=1$ and $s''=p-2$, there is just 1 family.

Summarizing these findings, we have the proposition below.

Proposition 6.2. *Let $f=1$. If $V_{\vec{t'},\vec{s'}}$ and $V_{\vec{t''},\vec{s''}}$ are a pair of non-isomorphic, non-Steinberg Serre weights, then a type II intersection occurs for the pair if and only if $e>1$, $p>3$ and $\mathrm{Ext}^1_{\mathrm{GL}_2(k)}(V_{\vec{t'},\vec{s'}},V_{\vec{t''},\vec{s''}})\neq 0$ via Proposition 2.1(ii)(b). In addition, the following statements are true:*

- *If (χ_1,χ_2) witness the type II intersection, then one of the two corresponding type I intersections witnessed by (χ_1,χ_2) arises via Proposition 5.2(i). The other arises via Proposition 5.2(ii).*
- *Each type II intersection is witnessed by 2 separate families except when $s'=1$ and $s''=p-2$, in which case just one family witnesses it.*

6.3. **Type II intersections when $f>1$, $e=1$.** We will use the algorithm in [DDR] for this section. Our objective is to find the conditions on $V_{\vec{t'},\vec{s'}}$ and $V_{\vec{t''},\vec{s''}}$ so that $\mu(J'_{max})=\mu(J''_{max})$ of cardinality $f-1$, where J'_{max} is the subset of $\mathbb{Z}/f\mathbb{Z}$ satisfying the conditions in Definition 4.11 for $V_{\vec{t'},\vec{s'}}$ while J''_{max} is the corresponding subset for $V_{\vec{t''},\vec{s''}}$.

We will find these intersections in two steps. First, we will find $V_{\vec{t'},\vec{s'}}$ and $V_{\vec{t''},\vec{s''}}$ such that $J'_{max}=\mathbb{Z}/f\mathbb{Z}-\{[f-1]\}$, $J''_{max}=\mathbb{Z}/f\mathbb{Z}-\{[f-1-i]\}$ for some $i\in[0,f-1]$, and $\omega_{f-1}^{-s'_{f-1}-1}\prod_{j\in T\smallsetminus\{f-1\}}\omega_j^{s'_j+1}=\omega_i^{-s''_{f-1-i}-1}\prod_{j\in T\smallsetminus\{f-1-i\}}\omega_j^{s''_j+1}$. The assumption that $J'_{max}=\mathbb{Z}/f\mathbb{Z}-\{[f-1]\}$ does not cause any loss of generality. In the second step, we will compute $\mu(J'_{max})$ and $\mu(J''_{max})$, and identify the situations in which they are the same.

For the first step, we will use the results of Proposition 5.4. Specifically, if a $V_{\vec{t'},\vec{s'}}$ exists with $J'_{max}=\mathbb{Z}/f\mathbb{Z}-\{[f-1]\}$, then there exists a non-Steinberg $V_{\vec{t},\vec{s}}$ so that the pair $(\vec{s},\vec{s'})$ satisfies one of the conditions enumerated in Proposition 5.4. This is simply a consequence of Lemma 4.21. Similarly, we can find a non-Steinberg $V_{\vec{d},\vec{\tilde{s}}}$ so that the pair $(\vec{\tilde{s}}, \vec{s''})$ satisfies one of the conditions enumerated in Proposition 5.4 after adding i to each index. Since we are imposing $\prod_{j\in T}\omega_j^{s_j} = \omega_{f-1}^{-s'_{f-1}-2}\prod_{j\in T\smallsetminus\{f-1\}}\omega_j^{s'_j} = \omega_i^{-s''_{f-1-i}-2}\prod_{j\in T\smallsetminus\{f-1-i\}}\omega_j^{s''_j+2} = \prod_{j\in T}\omega_j^{\tilde{s}_j}$, we have $\vec{s}=\vec{\tilde{s}}$. Therefore, in the first step we are looking for vectors $\vec{s}$ that show up in more than one items of the list in Proposition 5.4 (after translating the indices by adding some fixed integer if necessary). If such a $\vec{s}$ exits, we will say that the two items in the list can be *cycled* with each other, and that one item is a *cycling* of the other. The corresponding two $\vec{s'}$'s (in the notation of the list in Proposition 5.4) give us our candidate ($\vec{s'}$, $\vec{s''}$) and the reindexing informs us what i should be. In this situation, since $\chi_2^{-1}\chi_1$ must equal $\prod_{j\in T}\omega_i^{s_i+1}$, the tame signature $(a_{f-1},...,a_0) = (s_{f-1}+1,...,s_0+1)$.

For instance, consider the f-tuple $\vec{x}$ with $(x_{f-1}, x_{f-2}, ..., x_{f-m}, x_{f-1-m}) = (\in [0,p-2], p-1, ..., p-1, \in[0,p-2])$ for some $m\in[1,f-4]$, $(x_{f-1-i}, x_{f-2-i}, ..., x_{f-i-k}) = (p-1,p-1,...,p-1)$ for some $i\in[m+1,f-3]$, $k\in[2,f-1-i]$ and $x_{f-1-i-k}\in[0,p-2]$. Clearly, $\vec{x}$ satisfies the conditions required of $\vec{s}$ showing up in Proposition 5.4(i). If we reindex $\vec{x}$, adding i to the indices mod f, then we see that $\vec{x}$ can show up as the $\vec{s}$ in Proposition 5.4(v). That is, Proposition 5.4(i) can be cycled with Proposition 5.4(v). The corresponding two $\vec{s'}$ (in the notation of Proposition 5.4) that show up in Proposition 5.4(i) and Proposition 5.4(v) are our candidates for $\vec{s'}$ and $\vec{s''}$ respectively (in the notation of this proposition). The reindexing tells us that J'_{max} ought to be $\mathbb{Z}/f\mathbb{Z}\smallsetminus\{[f-1]\}$ and J''_{max} ought to be $\mathbb{Z}/f\mathbb{Z}\smallsetminus\{[f-1-i]\}$.

For the second step, we note that $\delta(f-2-i) = f-1-i$. Similarly, $\delta(f-3-i) = f-2-i$ and so on until $\delta(f-i-k) = f-i-k+1$. Therefore $f-i-k\notin\mu(J''_{max})$, which implies that $\mu(J''_{max}) = \mathbb{Z}/f\mathbb{Z}\smallsetminus\{[f-i-k]\}$. Similarly, $\delta(f-2) = f-1$ and if $m>1$, we observe that δ causes an increase in index right until $f-m$, so that $\delta(f-m) = f+1-m$. This forces $f-1-m\in\mu(J'_{max})$ and eventually, $f-i-k\in\mu(J'_{max})$. If $m=1$, $f-2-m$ is in $\mu(J'_{max})$ again forcing $f-i-k\in\mu(J'_{max})$. Hence $\mu(J'_{max})\neq\mu(J''_{max})$.

We repeat this process by finding all possible cyclings and computing $\mu(J'_{max})$ and $\mu(J''_{max})$. Instead of showing details for all computations, we will give an outline. Each $\vec{s}$ showing up in the list items of Proposition 5.4 has constraints for the components positioned in some specific way relative to the indices $f-1$ and $f-1-m$ for some m (In the notation of Proposition 5.4, the symbol i is used instead of m. Here we are using i differently, to indicate the translation of indices). If this $\vec{s}$ shows up in another list item after reindexing by adding i mod f, the constraints for the reindexed second list item will have a description relative to indices $f-1$ and $f-1-n$ for some n. After undoing the reindexing, we may expect to see constraints on $\vec{s}$ components positioned in a specific away around the indices given by $f-1$ and $f-1-m$ (as posed by the specifications of the first list item), and $f-1-i$ and $f-1-i-n$ (as posed by the specifications of the second list item). We will use this notation in the outline below.

(1) Proposition 5.4(i) can be cycled with Proposition 5.4(iv) in the following possible ways:
   (a) $i\in[m+1,f-2]$. Then $\mu(J''_{max})$ excludes $f-1-i$. If $m>1$, $\mu(J'_{max})$ excludes $f-m$. If $m=1$, $\mu(J'_{max})$ excludes $f-1$. Therefore, $\mu(J''_{max})\neq\mu(J'_{max})$.

   (b) $i = m-1 \geq 1$. Again, $\mu(J''_{max})$ excludes $f-1-i = f-m$. The same holds true for $\mu(J'_{max})$ and we have $\mu(J''_{max}) = \mu(J'_{max})$.
(2) Proposition 5.4(i) can be cycled with Proposition 5.4(v) in the following possible ways:
   (a) $i \in [m+1, f-3]$ and $f-1-i-n \not\equiv f-1 \mod f$. The calculations in the example above show that $\mu(J'_{max}) \neq \mu(J''_{max})$.
   (b) $i \in [m+1, f-3]$ and $f-1-i-n \equiv f-1 \mod f$. Here, $\mu(J''_{max})$ excludes 0, whereas $\mu(J'_{max})$ includes it, making them unequal.
   (c) $i \in [1, m-2]$ and $f-1-i-n = f-1-m$. Here $\mu(J'_{max})$ and $\mu(J''_{max})$ are both of cardinality $f-1$ and exclude $f-m$. Therefore, they are equal.
(3) Proposition 5.4(ii) can be cycled with Proposition 5.4(iv) with $i = f-1$. $\mu(J''_{max})$ excludes $f-1-i = 0$. The same is true for $\mu(J'_{max})$, which is thus equal to $\mu(J''_{max})$.
(4) Proposition 5.4(ii) can be cycled with Proposition 5.4(v) with any $i \in [1, f-2]$. In this case, both $\mu(J'_{max})$ and $\mu(J''_{max})$ exclude 0. They are therefore equal.
(5) Proposition 5.4(iii) can be cycled with Proposition 5.4(v) with $i \in [0, f-1] \setminus \{1\}$. Both $\mu(J'_{max})$ and $\mu(J''_{max})$ exclude $f-1$, and are equal.
(6) Proposition 5.4(iii) can be cycled with Proposition 5.4(vi) with $i = 1$. Both $\mu(J'_{max})$ and $\mu(J''_{max})$ exclude $f-1$, and are equal.
(7) Proposition 5.4(iii) can be cycled with Proposition 5.4(vii) with $i = 1$. Both $\mu(J'_{max})$ and $\mu(J''_{max})$ exclude $f-1$, and are equal.
(8) Proposition 5.4(iv) can be cycled with Proposition 5.4(v):
   (a) $i > 1$, $f-1-i-n = f-2$. Both $\mu(J'_{max})$ and $\mu(J''_{max})$ exclude $f-1$ and are equal.
   (b) $i > 1$, $f-1-i-n \neq f-2$. $\mu(J'_{max})$ exlucdes $f-1$, while $\mu(J''_{max})$ excludes $f-i-n$. Therefore, $\mu(J'_{max}) \neq \mu(J''_{max})$.
(9) Proposition 5.4(v) can be cycled with Proposition 5.4(vii) with $i = m$. Both $\mu(J'_{max})$ and $\mu(J''_{max})$ exclude $f-i$ and are equal.

Proposition 6.4. *Let $f > 1$, $e = 1$. There exists a pair of G_K characters (χ_1, χ_2) of highest weight $V_{\vec{t},\vec{s}}$ witnessing a type II intersection for $V_{\vec{t'},\vec{s'}}$ and $V_{\vec{t''},\vec{s''}}$ if and only if after translating the indices by adding some fixed integer, there exists an $i \in T$ such that the following are true:*

(i) $s'_{f-1}+1+\sum_{j=0}^{f-1} p^{f-1-j} t'_j \equiv p^i(s''_{f-1-i}+1)+\sum_{j=0}^{f-1} p^{f-1-j} d''_j \equiv \sum_{j=0}^{f-1} p^{f-1-j} t_j \mod p^f - 1$.

(ii) The vectors $\vec{s'}$, $\vec{s''}$ and $\vec{s}$ satisfy one of the following conditions:

   (a) $(s'_{f-1}, s'_{f-2}, ..., s'_{f-1-i}, s'_{f-2-i}) = (\in [0, p-2], 0, ..., 0, \in [1, p-2])$ *for some $i \in [1, f-2]$;*

   $(s''_{f-1}, s''_{f-2}, ..., s''_{f-1-i}, s''_{f-2-i}) = (p - s'_{f-1} - 2, p-1, ..., p-1, s'_{f-2-i} + 1)$*;*

   $(s_{f-1}, s_{f-2}, ..., s_{f-1-i}, s_{f-2-i}) = (p - s'_{f-1} - 2, p-1, ..., p-1, s'_{f-2-i} - 1)$*.*

(b) $f > 2$ *and* $(s'_{f-1}, s'_{f-2}, ..., s'_{f-1-i}, s'_{f-2-i}, s'_{f-3-i}, ..., s'_{f-m}, s'_{f-1-m}) = (\in [0, p-2], 0, ..., 0, 0, 0, ..., 0, \in [1, p-1])$ *for some* $m \in [3, f-1]$;

$(s''_{f-1}, s''_{f-2}, ..., s''_{f-1-i}, s''_{f-2-i}, s''_{f-3-i}, ..., s''_{f-m}, s''_{f-1-m}) = (p - s'_{f-1} - 2, p-1, ..., p-1, 1, 0, ..., 0, s'_{f-1-m})$ *where* $i \in [1, m-2]$;

$(s_{f-1}, s_{f-2}, ..., s_{f-1-i}, s_{f-2-i}, s_{f-3-i}, ..., s_{f-m}, s_{f-1-m}) = (p - s'_{f-1} - 2, p-1, ..., p-1, p-1, p-1, ..., p-1, s'_{f-1-m} - 1)$.

(c) $i = f-1$ *and* $(s'_{f-1}, s'_{f-2}, ..., s'_1, s'_0) = (\in [0, p-3], 0, ..., 0, 0)$;

$(s''_{f-1}, s''_{f-2}, ..., s''_1, s''_0) = (p - 1 - s'_{f-1}, p-1, ..., p-1, p-1)$;

$(s_{f-1}, s_{f-2}, ..., s_1, s_0) = (p - 3 - s'_{f-1}, p-1, ..., p-1, p-1)$.

(d) $f > 2$ *and* $(s'_{f-1}, s'_{f-2}, ..., s'_{f-1-i}, s'_{f-2-i}, s'_{f-3-i}, ..., s'_0) = (\in [0, p-3], 0, ..., 0, 0, 0, ..., 0)$;

$(s''_{f-1}, s''_{f-2}, ..., s''_{f-1-i}, s''_{f-2-i}, s''_{f-3-i}, ..., s''_0) = (p - 2 - s'_{f-1}, p-1, ..., p-1, 1, 0, ..., 0)$ *where* $i \in [1, f-2]$;

$(s_{f-1}, s_{f-2}, ..., s_{f-1-i}, s_{f-2-i}, s_{f-3-i}, ..., s_0) = (p - 3 - s'_{f-1}, p-1, ..., p-1, p-1, p-1, ..., p-1)$.

(e) $f > 2$ *and* $(s'_{f-1}, s'_{f-2}, ..., s'_{f-i}, s'_{f-1-i}, s'_{f-2-i}, s'_{f-3-i}, ..., s'_0) = (p-2, 0, ..., 0, 0, 0, 0, ..., 0)$;

$(s''_{f-1}, s''_{f-2}, ..., s''_{f-i}, s''_{f-1-i}, s''_{f-2-i}, s''_{f-3-i}, ..., s''_0) = (0, p-1, ..., p-1, p-1, 1, 0, ..., 0)$ *where* $i \in [2, f-1]$;

$(s_{f-1}, s_{f-2}, s_{f-3}, ..., s_0) = (p-1, p-2, p-1, ..., p-1)$.

(f) $f = 2$ *and* $(s'_{f-1}, s'_{f-2}) = (p-2, 0)$;

$(s''_{f-1}, s''_{f-2}) = (1, p-1)$ *where* $i = 1$;

$(s_{f-1}, s_{f-2}) = (p-1, p-2)$.

(g) $f > 2$ *and* $(s'_{f-1}, s'_{f-2}, s'_{f-3}, s'_{f-4}, ..., s'_0) = (p-2, 0, 0, 0, ..., 0)$;

$(s''_{f-1}, s''_{f-2}, s''_{f-3}, ..., s''_0) = (0, p-1, 1, 0, ..., 0)$ *where* $i = 1$;

$(s_{f-1}, s_{f-2}, s_{f-3}, s_{f-4} ..., s_0) = (p-1, p-2, p-1, p-1, ..., p-1)$.

(h) $f > 2$ *and* $(s'_{f-1}, s'_{f-2}, s'_{f-3}, ..., s'_{f-1-i}, s'_{f-2-i})$
$= (p-1, 1, 0, ..., 0, \in [1, p-2])$ *where* $i > 1$*;*

$(s''_{f-1}, s''_{f-2}, s''_{f-3}, ..., s''_{f-1-i}, s''_{f-2-i}) =$
$(p-1, p-1, p-1, ..., p-1, s'_{f-2-i} + 1)$*;*

$(s_{f-1}, s_{f-2}, s_{f-3}, ..., s_{f-1-i}, s_{f-2-i}) =$
$(p-1, p-1, p-1, ..., p-1, s'_{f-2-i} - 1)$.

(i) $f > 2$, $i = f-1$ *and* $(s'_{f-1}, s'_{f-2}, s'_{f-3}, ..., s'_1, s'_0) =$
$(p-1, 1, 0, ..., 0, p-1)$*;*

$(s''_{f-1}, s''_{f-2}, s''_{f-3}, ..., s''_1, s''_0) = (1, 0, 0, ..., 0, p-1)$*;*

$(s_{f-1}, s_{f-2}, s_{f-3}, ..., s_1, s_0) = (p-1, p-1, p-1, ..., p-1, p-2)$.

Proof. The conditions on $\vec{s'}$, $\vec{s''}$ and $\vec{s}$ are a consequence of the preceding discussion along with explicit descriptions coming from the list in Proposition 5.4. The condition on $\vec{t'}$, $\vec{t''}$ and $\vec{t}$ follow from comparing descriptions of χ_2 using Lemmas 4.18 and 4.21. □

Remark 6.5. In each triple of $\vec{s'}$, $\vec{s''}$ and $\vec{s}$ featuring in the list in Proposition 6.4, at least two of the vectors have some component equal to $p-1$.

6.6. **Type II intersections when $f > 1$, $e > 1$.** We will compute the scenarios in which type II intersections occur for the pair $V_{\vec{t'},\vec{s'}}$ and $V_{\vec{t''},\vec{s''}}$. In the case of $V_{\vec{t'},\vec{s'}}$, we will assume without loss of generality that $i = f-1$ in the statements of Lemma 4.21 and that $\vec{t'} = 0$.

In the following calculations, we will use some extra notation and strategies for comparing $J^{AH}_{V_{\vec{t'},\vec{s'}}}(\chi_1, \chi_2)$ and $J^{AH}_{V_{\vec{t''},\vec{s''}}}(\chi_1, \chi_2)$ that we now explain. Given the Serre weights $V_{\vec{t'},\vec{s'}}$ and $V_{\vec{t''},\vec{s''}}$, and suitable G_K characters χ_1 and χ_2, we may compute $y'_j, y''_j, z'_j, z''_j, \lambda'_j, \lambda''_j, \mathcal{I}'_j, \mathcal{I}''_j, \xi'_j$ and ξ''_j using Definitions 4.3, 4.5 and 4.9.

Definition 6.7. Fix $j \in T$. $V'_j \subset \mathbb{Z}$ is defined to satisfy:

$$\{\xi'_j - u(p^f - 1) | u \in \mathcal{I}'_j\} = \{(p^f - 1)v + \lambda'_j | v \in V'_j\}$$

$V''_j \subset \mathbb{Z}$ is defined to satisfy:

$$\{\xi''_j - u(p^f - 1) | u \in \mathcal{I}''_j\} = \{(p^f - 1)v + \lambda'_j | v \in V''_j\}$$

The above definition of V''_j makes sense because $\lambda''_j \equiv \lambda'_j \mod p^{f-1}$, since exponentiating ω_j with either gives the same character $\chi_2^{-1}\chi_1$.

Definition 6.8. Define $P', P'' \subset T \times \mathbb{Z}$ as follows:

$$P' := \{(j, v) \in T \times \mathbb{Z} | v \in V'_j\}$$
$$P'' := \{(j, v) \in T \times \mathbb{Z} | v \in V''_j\}$$

We define two functions β and α next.

Definition 6.9.

$$\begin{aligned}\beta : T \times \mathbb{Z} &\to \mathbb{Z}\\ (j,v) &\mapsto (p^f-1)v + \lambda'_j\end{aligned}$$

and,

$$\begin{aligned}\alpha : T \times \mathbb{Z} &\to \mathbb{Z} \times \{0,1,...,f''-1\}\\ (j,v) &\mapsto (m,\kappa)\end{aligned}$$

where $m = \dfrac{\beta(j,v)}{p^{\mathrm{val}_p(\beta(j,v))}}$, and κ satisfies (4.10.2).

Remark 6.10. By Definitions 4.10 and 6.9, $J^{AH}_{V_{\vec{t'},\vec{s'}}}(\chi_1,\chi_2) = \{\alpha(j,v) | (j,v) \in P'\}$ and $J^{AH}_{V_{\vec{t''},\vec{s''}}}(\chi_1,\chi_2) = \{\alpha(j,v) | (j,v) \in P''\}$.

Remark 6.11. By the comments following Definition 4.10, $\alpha|_{P'}$ and $\alpha|_{P''}$ are injective functions.

Remark 6.12. An examination of Definition 4.10 shows that if $v \in V'_j$ for some $j \in T$, then finding a pair $(\tilde{j},\tilde{v}) \in P''$ such that $\alpha(j,v) = \alpha(\tilde{j},\tilde{v})$ is equivalent to finding $\tilde{j}$ and $\tilde{v} \in V''_{\tilde{j}}$ satisfying the following two conditions:

- $(p^f-1)v - \lambda'_j$ and $(p^f-1)\tilde{v} - \lambda'_j$ differ by a factor of a p-power;
- the difference of p-adic valuations offsets the difference between j and $\tilde{j}$ in the formula for computing κ in (4.10.2).

Remark 6.13. Let $\alpha(j,v) = \alpha(\tilde{j},\tilde{v})$. Then $j = \tilde{j} \iff \mathrm{val}_p((p^f-1)v - \lambda'_j) \equiv \mathrm{val}_p((p^f-1)\tilde{v} - \lambda'_{\tilde{j}}) \mod f$.

Remark 6.14. If $j \neq \tilde{j}$ and $\mathrm{val}_p((p^f-1)v - \lambda'_j) = \mathrm{val}_p((p^f-1)\tilde{v} - \lambda'_{\tilde{j}}) = 0$, then $\alpha(j,v) \neq \alpha(\tilde{j},\tilde{v})$.

Definition 6.15. We will say that a pair $(j,v) \in P'$ *matches* $(\tilde{j},\tilde{v}) \in P''$ if $\alpha(j,v) = \alpha(\tilde{j},\tilde{v})$.

Remark 6.16. For the purposes of our calculations, we will classify the ways a pair $(j,v) \in P'$ can match a pair $(\tilde{j},\tilde{v}) \in P''$ in the following manner:

(i) $(j,v) = (\tilde{j},\tilde{v})$.
(ii) $\tilde{j} \equiv j+1 \mod f$ and $\tilde{v} = pv + z'_{j+1} - y'_{j+1}$; or $j \equiv \tilde{j}+1$ and $v = p\tilde{v} + z'_j - y'_j$. In these cases, $|\mathrm{val}_p((p^f-1)v - \lambda'_j) - \mathrm{val}_p((p^f-1)\tilde{v} - \lambda'_{\tilde{j}})| = 1$.
(iii) Matches not classified by either of the above.

As we will see, the first two types will be easy to spot, whereas the third will need some verification.

We will use the notation and ideas above repeatedly in the calculations below. Because of the repetitiveness of the arguments, we will show the calculations in detail only for a few scenarios, and will only report the findings from the calculations for the rest.

6.16.1. *Case 1.* : $|J^{AH}_{V_{\vec{t'},\vec{s'}}}(\chi_1,\chi_2)| = |J^{AH}_{V_{\vec{t''},\vec{s''}}}(\chi_1,\chi_2)| = ef-1$, both via Lemma 4.21(i).

Case 1a. : $i = f-1$. Comparing ways of writing $\chi_2^{-1}\chi_1$ and χ_2 in terms of $\vec{s'}$, $\vec{s''}$ and $\vec{t''}$ using Remark 4.22, we obtain that $V_{\vec{t'},\vec{s'}} = V_{\vec{t''},\vec{s''}}$, a contradiction.

Case 1b. : $i < f-1$.

Comparing ways of writing $\chi_2^{-1}\chi_1$ in terms of $\vec{s'}$ and $\vec{s'}$, we have:

$$e-2-s'_{f-1}+\sum_{j\in T\smallsetminus\{f-1\}} p^{f-1-j}(s'_j+e) \equiv$$
$$p^{f-1-i}(e-2-s''_i)+\sum_{j\in T\smallsetminus\{i\}} p^{f-1-j}(s''_j+e)$$

$$\iff \quad -2-s'_{f-1}+\sum_{j\in T\smallsetminus\{f-1\}} p^{f-1-j}s'_j \equiv p^{f-1-i}(-2-s''_i)+\sum_{j\in T\smallsetminus\{i\}} p^{f-1-j}s''_j$$

$$\iff \quad p-2-s'_{f-1}+p(s'_{f-2}-1)+\sum_{j\in T\smallsetminus\{f-1,f-2\}} p^{f-1-j}s'_j \equiv$$
$$p^{f-1-i}(p-2-s''_i)+p^{f-i}(s''_{i-1}-1)\sum_{j\in T\smallsetminus\{i,i-1\}} p^{f-1-j}s''_j V_{\vec{t'},\vec{s'}}$$
$$\iff \quad (p-2-s'_{f-1},s'_{f-2}-1,s'_{f-3},...,s'_0)\equiv$$
$$(s''_{f-1},...,s''_{i+1},p-2-s''_i,s''_{i-1}-1,s''_{i-2}...,s''_0)$$

Lemma 6.17. *The above condition is satisfied if and only if (upto interchanging $\vec{s'}$ with $\vec{s''}$), one of the following pairs describe $\vec{s'}$ and $\vec{s''}$:*

(i) $(s'_{f-1},s'_{f-2},...,s'_{k+1},s'_k)=(\in[0,p-2],0,...,0,\in[1,p-1])$ *for some* $k\in[i+1,f-2]$,
$(s'_i,s'_{i-1},...,s'_{l+1},s'_l)=(\in[0,p-2],p-1,...,p-1,\in[0,p-2])$ *for some* $l\in[0,i-1]$*;*
$(s''_{f-1},s''_{f-2},...,s''_{k+1},s''_k)=(p-2-s'_{f-1},p-1,...,p-1,s'_k-1)$,
$(s''_i,s''_{i-1},...,s''_{l+1},s''_l)=(p-2-s'_i,0,...,0,s'_l+1)$.

(ii) $(s'_{f-1},s'_{f-2},...,s'_{k+1},s'_k)=(\in[0,p-2],0,...,0,\in[1,p-1])$ *for some* $k\in[i+1,f-2]$,
$(s'_i,s'_{i-1},...,s'_0)=(\in[0,p-2],p-1,...,p-1)$*;*
$(s''_{f-1},s''_{f-2},...,s''_{k+1},s''_k)=(p-1-s'_{f-1},p-1,...,p-1,s'_k-1)$,
$(s''_i,s''_{i-1},...,s''_0)=(p-2-s'_i,0,...,0)$.

(iii) $(s'_{f-1},s'_{f-2},...,s'_{i+1},s'_i,s'_{i-1},...,s'_0)=(\in[0,p-2],0,...,0,\in[1,p-1],p-1,...,p-1)$*;*
$(s''_{f-1},s''_{f-2},...,s''_{i+1},s''_i,s''_{i-1},...,s''_0)=(p-1-s'_{f-1},p-1,...,p-1,p-s'_i-1,0,...,0)$.

Proof. Easy verification upon recalling that $s'_{f-1},s''_i\le p-2$ by Lemma 4.21(i). □

Imposing the above conditions, we now calculate y'_j, y''_j, z'_j, z''_j, $\mathcal{I}'_j$, $\mathcal{I}''_j$, V'_j and V''_j, and compare $J^{AH}_{V_{\vec{t'},\vec{s'}}}(\chi_1,\chi_2)$ with $J^{AH}_{V_{\vec{t''},\vec{s''}}}(\chi_1,\chi_2)$ using Remark 6.10.

For Lemma 6.17(i), we have:

$$y'_j=\begin{cases} s'_j+1 & \text{if } j=f-1\\ 0 & \text{if } j\in T\smallsetminus\{f-1\}\end{cases} \tag{6.17.1}$$

$$y''_j = \begin{cases} p-1-s'_j & \text{if } j=i \\ 0 & \text{if } j \in T \smallsetminus \{i\} \end{cases} \tag{6.17.2}$$

$$z'_j = \begin{cases} e-1 & \text{if } j=f-1 \\ e+s'_j & \text{if } j \in T \smallsetminus \{f-1\} \end{cases} \tag{6.17.3}$$

$$z''_j = \begin{cases} p+e-2-s'_j = p+(z'_j-y'_j) & \text{if } j=f-1 \\ p+e-1 = p-1+(z'_j-y'_j) & \text{if } j \in [k+1,f-2] \\ e-1+s'_j = -1+(z'_j-y'_j) & \text{if } j=k \\ e-1 = -p+(z'_j-y'_j)+y''_j & \text{if } j=i \\ e = -(p-1)+(z'_j-y'_j) & \text{if } j \in [l+1,i-1] \\ e+1+s'_j = 1+(z'_j-y'_j) & \text{if } j=l \\ e+s'_j = (z'_j-y'_j) & \text{if } j \in [i+1,k-1] \cup [0,l-1] \end{cases} \tag{6.17.4}$$

$$\mathcal{I}'_j = \begin{cases} [0,e-2] & \text{if } j=f-1 \\ \{0\} \cup [s'_j+1, s'_j+e-1] & \text{if } j \in T \smallsetminus \{f-1\} \end{cases} \tag{6.17.5}$$

$$\mathcal{I}''_j = \begin{cases} \{0\} \cup [p-1-s'_j, z''_j-1] & \text{if } j=f-1 \\ \{0\} \cup [p, z''_j-1] & \text{if } j \in [k+1,f-2] \\ \{0\} \cup [s'_j, z''_j-1] & \text{if } j=k \\ [0,e-2] & \text{if } j=i \\ \{0\} \cup [1, z''_j-1] & \text{if } j \in [l+1,i-1] \\ \{0\} \cup [s''_j+2, z''_j-1] & \text{if } j=l \\ \{0\} \cup [s''_j, z''_j-1] & \text{if } j \in [i+1,k-1] \cup [0,l-1] \end{cases} \tag{6.17.6}$$

$$V'_j = \begin{cases} [1,e-1] & \text{if } j=f-1 \\ [1,e-1] \cup \{s'_j+e\} & \text{if } j \in T \smallsetminus \{f-1\} \end{cases} \tag{6.17.7}$$

$$V''_j = \begin{cases} [1,e-1] \cup \{p+z'_j-y'_j\} & \text{if } j=f-1 \\ [2,e] \cup \{p+z'_j-y'_j\} & \text{if } j \in [k+1,f-2] \\ [2,e] \cup \{s'_j+e\} & \text{if } j=k \\ [1,e-1] & \text{if } j=i \\ [0,e-1] & \text{if } j \in [l+1,i-1] \\ [0,e-2] \cup \{s'_j+e\} & \text{if } j=l \\ [1,e-1] \cup \{s'_j+e\} & \text{if } j \in [i+1,k-1] \cup [0,l-1] \end{cases} \tag{6.17.8}$$

Recall that $J^{AH}_{V_{\vec{t'},\vec{s'}}}(\chi_1,\chi_2) = \{\alpha(j,v) | (j,v) \in P'\}$ and $J^{AH}_{V_{\vec{t''},\vec{s''}}}(\chi_1,\chi_2) = \{\alpha(j,v) | (j,v) \in P''\}$. In order to compare the two, there is no work to be done for $(j,v) \in P' \cap P''$. So, we must now examine the image of α when restricted to the set $P' - P''$ and compare it to the image of α when restricted to the set $P'' - P'$.

To begin, consider $\{(j,1) | j \in [k,f-2]\} \subset P' - P''$. One can immediately verify using Remark 6.12 that $\alpha(j,1) = \alpha(j+1, p+z'_j-y'_j)$, where $(j,1) \in P' - P''$ and

$(j+1, p+z'_j-y'_j) \in P''-P'$. Similarly, for $j \in [l+1, i]$, $\alpha(j, z'_j-y'_j) = \alpha(j-1, 0)$. Here $(j, z'_j-y'_j) \in P'-P''$ (since $z'_j-y'_j = s'_j+e$) and $(j-1, 0) \in P''-P'$. These matches are of the type described in Remark 6.16(ii). After taking into account all matches of the types described in Remark 6.16(i) and Remark 6.16(ii), the only possibly unmatched pairs are $(l, e-1) \in P'-P''$ and $(k, e) \in P''-P'$. Now, $\mathrm{val}_p(\beta(l, e-1)) = 0$ as $s'_l \neq p-1$. As $s'_k \neq 1$, $\mathrm{val}_p(\beta(k, e)) = 0$. As $l \neq k$, $\alpha(l, e-1) \neq \alpha(k, e)$ by Remark 6.14.

Therefore, $J^{AH}_{V_{\vec{t'},\vec{s'}}}(\chi_1, \chi_2) \neq J^{AH}_{V_{\vec{t''},\vec{s''}}}(\chi_1, \chi_2)$.

Calculations for Lemma 6.17(ii) are as follows:

$$y'_j = \begin{cases} s'_j+1 & \text{if } j = f-1 \\ 0 & \text{if } j \in T \smallsetminus \{f-1\} \end{cases} \tag{6.17.9}$$

$$y''_j = \begin{cases} p-1-s'_j & \text{if } j = i \\ 0 & \text{if } j \in T \smallsetminus \{i\} \end{cases} \tag{6.17.10}$$

$$z'_j = \begin{cases} e-1 & \text{if } j = f-1 \\ e+s'_j & \text{if } j \in T \smallsetminus \{f-1\} \end{cases} \tag{6.17.11}$$

$$z''_j = \begin{cases} p+e-1-s'_j = p+(z'_j-y'_j)+1 & \text{if } j = f-1 \\ p+e-1 = p-1+(z'_j-y'_j) & \text{if } j \in [k+1, f-2] \\ e-1+s'_j = -1+(z'_j-y'_j) & \text{if } j = k \\ e+s'_j = (z'_j-y'_j) & \text{if } j \in [i+1, k-1] \\ e-1 = -p+(z'_j-y'_j)+y''_j & \text{if } j = i \\ e = -(p-1)+(z'_j-y'_j) & \text{if } j \in [0, i-1] \end{cases} \tag{6.17.12}$$

$$\mathcal{I}'_j = \begin{cases} [0, e-2] & \text{if } j = f-1 \\ \{0\} \cup [s'_j+1, s'_j+e-1] & \text{if } j \in T \smallsetminus \{f-1\} \end{cases} \tag{6.17.13}$$

$$\mathcal{I}''_j = \begin{cases} \{0\} \cup [p-s'_j, z''_j-1] & \text{if } j = f-1 \\ \{0\} \cup [p, z''_j-1] & \text{if } j \in [k+1, f-2] \\ \{0\} \cup [s'_j, z''_j-1] & \text{if } j = k \\ \{0\} \cup [s''_j+1, z''_j-1] & \text{if } j \in [i+1, k-1] \\ [0, e-2] & \text{if } j = i \\ \{0\} \cup [1, z''_j-1] & \text{if } j \in [0, i-1] \end{cases} \tag{6.17.14}$$

$$V'_j = \begin{cases} [1, e-1] & \text{if } j = f-1 \\ [1, e-1] \cup \{s'_j+e\} & \text{if } j \in T \smallsetminus \{f-1\} \end{cases} \tag{6.17.15}$$

$$V_j'' = \begin{cases} [0, e-2] \cup \{p + z_j' - y_j'\} & \text{if } j = f-1 \\ [2, e] \cup \{p + z_j' - y_j'\} & \text{if } j \in [k+1, f-2] \\ [2, e] \cup \{s_j' + e\} & \text{if } j = k \\ [1, e-1] \cup \{s_j' + e\} & \text{if } j \in [i+1, k-1] \\ [1, e-1] & \text{if } j = i \\ [0, e-1] & \text{if } j \in [0, i-1] \end{cases} \tag{6.17.16}$$

To verify $J^{AH}_{V_{\vec{t'},\vec{s'}}}(\chi_1, \chi_2) = J^{AH}_{V_{\vec{t''},\vec{s''}}}(\chi_1, \chi_2)$, we apply the same strategy as we previously did. As before, for each $(j, v) \in P$ except $(f-1, e-1)$, we can get (j, v) to match some $(\tilde{j}, \tilde{v})$ with $(\tilde{j}, \tilde{v}) \in P''$ via Remark 6.16(i) or Remark 6.16(ii). (k, e) is the only pair in P'' not matched to anything in $P - \{(f-1, e-1)\}$ via these two matching strategies. By Remark 6.14, $(f-1, e-1)$ cannot match (k, e) because $\mathrm{val}_p(\beta(f-1, e-1)) = 0 = \mathrm{val}_p(\beta(k, e))$, since $s'_{f-1} \neq p-1$ and $s'_k \neq 0$. Therefore, we do not get a type II intersection in the desired manner.

The calculations for Lemma 6.17(iii) are similar and left to the reader. The results from the calculations are also similar, and show that $J^{AH}_{V_{\vec{t'},\vec{s'}}}(\chi_1, \chi_2) \neq J^{AH}_{V_{\vec{t''},\vec{s''}}}(\chi_1, \chi_2)$.

The findings are summarized below.

Proposition 6.18. *Let $e > 1$, $f > 1$. Suppose $V_{\vec{t'},\vec{s'}}$ and $V_{\vec{t''},\vec{s''}}$ are a pair of non-isomorphic, non-Steinberg Serre weights. There do not exist any G_K characters χ_1 and χ_2 such that $|J^{AH}_{V_{\vec{t'},\vec{s'}}}(\chi_1, \chi_2)| = ef - 1$ via Lemma 4.21(i), $|J^{AH}_{V_{\vec{t''},\vec{s''}}}(\chi_1, \chi_2)| = ef - 1$ via Lemma 4.21(i) and $J^{AH}_{V_{\vec{t'},\vec{s'}}}(\chi_1, \chi_2) = J^{AH}_{V_{\vec{t''},\vec{s''}}}(\chi_1, \chi_2)$.*

6.18.1. *Case 2.* : $|J^{AH}_{V_{\vec{t'},\vec{s'}}}(\chi_1, \chi_2)| = ef - 1$ via Lemma 4.21(iii); $|J^{AH}_{V_{\vec{t''},\vec{s''}}}(\chi_1, \chi_2)| = ef - 1$ via Lemma 4.21(iii).
Case 2a. : $i = f-1$. Comparing ways of writing $\chi_2^{-1}\chi_1$ and χ_2 in terms of $\vec{s'}$, $\vec{s''}$ and $\vec{t'}$ using Remark 4.22, we obtain that $V_{\vec{t'},\vec{s'}} = V_{\vec{t''},\vec{s''}}$, a contradiction.
Case 2b. : $i < f-1$.

Comparing ways of writing $\chi_2^{-1}\chi_1$ in terms of $\vec{s'}$, $\vec{s''}$ and $\vec{s}$, we have:

$$\begin{aligned} &e - 2 + s'_{f-1} + \sum_{j \in T \smallsetminus \{f-1\}} p^{f-1-j}(s'_j + e) \\ &\qquad \equiv p^{f-1-i}(e - 2 + s''_i) + \sum_{j \in T \smallsetminus \{i\}} p^{f-1-j}(s''_j + e) \equiv \sum_{j \in T} p^{f-1-j}(s_j + e) \end{aligned}$$

(6.18.1)

$$\begin{aligned} \iff (-2 + s'_{f-1}, s'_{f-2}, ..., s'_0) &\equiv (s''_{f-1}, ..., s''_{i+1}, -2 + s''_i, s''_{i-1}, ..., s''_0) \\ &\equiv (s_{f-1}, s_{f-2}, ..., s_0) \end{aligned}$$

Comparing ways of writing χ_2, we have:

$$p^{f-1-i} + \sum_{j \in T} p^{f-1-j} d''_j \equiv 1 \equiv \sum_{j \in T} p^{f-1-j} t_j \mod p^f - 1$$

$$\iff \sum_{j \in T} p^{f-1-j} d''_j \equiv 1 - p^{f-1-i}, \quad \sum_{j \in T} p^{f-1-j} t_j \equiv 1 \mod p^f - 1 \tag{6.18.2}$$

Lemma 6.19. *The condition in (6.18.1) is satisfied for some $\vec{s'}$, $\vec{s''}$ and $\vec{s}$ if and only if one of the following pairs describe $\vec{s'}$ and $\vec{s''}$:*

(i) $s'_{f-1} \in [2, p-1]$, $s'_i \in [0, p-3]$;
$s''_{f-1} = s'_{f-1} - 2$, $s''_i = s'_i + 2$.

(ii) $(s'_{f-1}, s'_{f-2}, ..., s'_{k+1}, s'_k) = (\in [0,1], 0, ..., 0, \in [1, p-1])$ *for some* $k \in [i+1, f-2]$, $s'_i \in [0, p-3]$;
$(s''_{f-1}, s''_{f-2}, ..., s''_{k+1}, s''_k) = (p-2+s'_{f-1}, p-1, ..., p-1, s'_k - 1)$, $s''_i = s'_i + 2$.

(iii) $(s'_{f-1}, s'_{f-2}, ..., s'_{i+1}, s'_i) = (1, 0, ..., 0, \in [1, p-2])$;
$(s''_{f-1}, s''_{f-2}, ..., s''_{i+1}, s''_i) = (p-1, p-1, ..., p-1, s'_i + 1)$.

(iv) $s'_{f-1} \in [2, p-1]$, $(s'_i, s'_{i-1} ..., s'_{l+1}, s'_l) = (p-1, p-1, ..., p-1, \in [0, p-2])$ *for some* $l \in [0, i-1]$;
$s''_{f-1} = s'_{f-1} - 2$, $(s''_i, s''_{i-1} ..., s''_{l+1}, s''_l) = (1, 0, ..., 0, s'_l + 1)$.

(v) $s'_{f-1} \in [1, p-1]$, $(s'_i, s'_{i-1} ..., s'_0) = (p-1, p-1, ..., p-1)$;
$s''_{f-1} = s'_{f-1} - 1$, $(s''_i, s''_{i-1} ..., s''_0) = (1, 0, ..., 0)$;
$s_{f-1} \in [1, p-1]$, $(s_i, s_{i-1} ..., s_0) = (p-1, p-1, ..., p-1)$.

(vi) $(s'_{f-1}, s'_{f-2}, ..., s'_{k+1}, s'_k) = (1, 0, ..., 0, \in [1, p-1])$ *for some* $k \in [i+1, f-2]$,
$(s'_i, s'_{i-1}, ..., s'_{l+1}, s'_l) = (p-1, p-1, ..., p-1, \in [0, p-2])$ *for some* $l \in [0, i-1]$;
$(s''_{f-1}, s''_{f-2}, ..., s''_{k+1}, s''_k) = (p-2+s'_{f-1}, p-1, ..., p-1, s'_k - 1)$,
$(s''_i, s''_{i-1}, ..., s''_{l+1}, s''_l) = (1, 0, ..., 0, s'_l + 1)$.

Proof. Easy verification upon recalling that $s'_{f-1}, s''_i \geq 1$ byLemma 4.21(iii). □

Imposing the above conditions, we now calculate y'_j, y''_j, z'_j, z''_j, $\mathcal{I}'_j$, $\mathcal{I}''_j$, V'_j and V''_j, and compare $J^{AH}_{V_{\vec{t'},\vec{s'}}}(\chi_1, \chi_2)$ with $J^{AH}_{V_{\vec{t''},\vec{s''}}}(\chi_1, \chi_2)$ using Remark 6.10.

For Lemma 6.19(i), we have:

(6.19.1)
$$y'_j = \begin{cases} 1 & \text{if } j = f-1 \\ 0 & \text{if } j \in T \setminus \{f-1\} \end{cases}$$

(6.19.2)
$$y''_j = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \in T \setminus \{i\} \end{cases}$$

(6.19.3)
$$z'_j = \begin{cases} e-1+s'_j & \text{if } j = f-1 \\ e+s'_j & \text{if } j \in T \setminus \{f-1\} \end{cases}$$

(6.19.4)
$$z''_j = \begin{cases} e-2+s'_j = z'_j - y'_j & \text{if } j = f-1 \\ e+1+s'_i = (z'_j - y'_j) + y''_j & \text{if } j = i \\ e+s'_j = z'_j - y'_j & \text{if } j \in T \setminus \{f-1, i\} \end{cases}$$

(6.19.5)
$$\mathcal{I}'_j = \begin{cases} \{1\} \cup [s'_j + 1, z'_j - 1] & \text{if } j = f-1 \\ \{0\} \cup [s'_j + 1, z'_j - 1] & \text{if } j \in T \setminus \{f-1\} \end{cases}$$

$$\mathcal{I}''_j = \begin{cases} \{0\} \cup [s'_j - 1, z''_j - 1] & \text{if } j = f-1 \\ \{1\} \cup [s'_i + 3, z''_j - 1] & \text{if } j = i \\ \{0\} \cup [s'_j + 1, z'_j - 1] & \text{if } j \in T \smallsetminus \{f-1, i\} \end{cases} \tag{6.19.6}$$

$$V'_j = \begin{cases} [1, e-2] \cup \{z'_j - y'_j\} & \text{if } j = f-1 \\ [1, e-1] \cup \{z'_j - y'_j\} & \text{if } j \in T \smallsetminus \{f-1\} \end{cases} \tag{6.19.7}$$

$$V''_j = \begin{cases} [1, e-1] \cup \{z'_j - y'_j\} & \text{if } j = f-1 \\ [1, e-2] \cup \{z'_j - y'_j\} & \text{if } j = i \\ [1, e-1] \cup \{z'_j - y'_j\} & \text{if } j \in T \smallsetminus \{f-1, i\} \end{cases} \tag{6.19.8}$$

The only pairs in P and P'' that are unmatched after applying matching strategy Remark 6.16(i) are $(i, e-1) \in P$ and $(f-1, e-1) \in P''$. As $s'_i \neq p-1$ and $s'_{f-1} \neq 1$, $\mathrm{val}_p(\beta(i, e-1)) = 0 = \mathrm{val}_p(\beta(f-1, e-1))$. By Remark 6.14, $(i, e-1)$ cannot possibly match $(f-1, e-1)$, and therefore, $J^{AH}_{V_{\vec{t'}, \vec{s'}}}(\chi_1, \chi_2) \neq J^{AH}_{V_{\vec{t''}, \vec{s''}}}(\chi_1, \chi_2)$.

For Lemma 6.19(ii), we have:

$$y'_j = \begin{cases} 1 & \text{if } j = f-1 \\ 0 & \text{if } j \in T \smallsetminus \{f-1\} \end{cases} \tag{6.19.9}$$

$$y''_j = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \in T \smallsetminus \{i\} \end{cases} \tag{6.19.10}$$

$$z'_j = \begin{cases} e - 1 + s'_j & \text{if } j = f-1 \\ e + s'_j & \text{if } j \in T \smallsetminus \{f-1\} \end{cases} \tag{6.19.11}$$

$$z''_j = \begin{cases} p + e - 2 + s'_j = p + (z'_j - y'_j) & \text{if } j = f-1 \\ p + e - 1 = (p-1) + (z'_j - y'_j) & \text{if } j \in [k+1, f-2] \\ e - 1 + s'_j = -1 + (z'_j - y'_j) & \text{if } j = k \\ e + 1 + s'_i = (z'_j - y'_j) + y''_j & \text{if } j = i \\ e + s'_j = z'_j - y'_j & \text{if } j \notin \{i\} \cup [k, f-1] \end{cases} \tag{6.19.12}$$

$$\mathcal{I}'_j = \begin{cases} \{1\} \cup [s'_j + 1, z'_j - 1] & \text{if } j = f-1 \\ \{0\} \cup [s'_j + 1, z'_j - 1] & \text{if } j \in T \smallsetminus \{f-1\} \end{cases} \tag{6.19.13}$$

$$\mathcal{I}''_j = \begin{cases} \{0\} \cup [p + s'_j - 1, z''_j - 1] & \text{if } j = f-1 \\ \{0\} \cup [p, z''_j - 1] & \text{if } j \in [k+1, f-2] \\ \{0\} \cup [s'_j, z''_j - 1] & \text{if } j = k \\ \{1\} \cup [s'_i + 3, z''_j - 1] & \text{if } j = i \\ \{0\} \cup [s'_j + 1, z'_j - 1] & \text{if } j \notin \{i\} \cup [k, f-1] \end{cases} \tag{6.19.14}$$

$$V'_j = \begin{cases} [1, e-2] \cup \{z'_j - y'_j\} & \text{if } j = f-1 \\ [1, e-1] \cup \{z'_j - y'_j\} & \text{if } j \in T \smallsetminus \{f-1\} \end{cases} \tag{6.19.15}$$

$$V''_j = \begin{cases} [1,e-1]\cup\{p+z'_j-y'_j\} & \text{if } j=f-1 \\ [2,e]\cup\{p+z'_j-y'_j\} & \text{if } j\in[k+1,f-2] \\ [2,e]\cup\{z'_j-y'_j\} & \text{if } j=k \\ [1,e-2]\cup\{z'_j-y'_j\} & \text{if } j=i \\ [1,e-1]\cup\{z'_j-y'_j\} & \text{if } j\notin\{i\}\cup[k,f-1] \end{cases} \tag{6.19.16}$$

The only pairs in P and P'' that are unmatched after applying matching strategies in Remark 6.16(i) and Remark 6.16(ii) are those in $\{(f-1,z'_{f-1}-y'_{f-1}),(i,e-1)\}\subset P$ and $\{(f-1,e-1),(k,e)\}\subset P''$. As $\mathrm{val}_p(\beta(i,e-1))=0=\mathrm{val}_p(\beta(k,e))$ and $i\neq k$, $J^{AH}_{V_{\vec{t'},\vec{s'}}}(\chi_1,\chi_2)=J^{AH}_{V_{\vec{t''},\vec{s''}}}(\chi_1,\chi_2)$ if and only if $(i,e-1)$ matches $(f-1,e-1)$, while $(f-1,z'_{f-1}-y'_{f-1})$ matches (k,e). Suppose this is true and $(m,\kappa)=\alpha(f-1,z'_{f-1}-y'_{f-1})=\alpha(k,e)$. Plugging this data into the formula for κ in (4.10.2), we get:

$$\mathrm{val}_p(\beta(f-1,z'_{f-1}-y'_{f-1}))\equiv f-1-k \mod f \tag{6.19.17}$$

Therefore, $p^{f-1-k}|\beta(f-1,z'_{f-1}-y'_{f-1})=p(z'_{f-2}-y'_{f-2})+p^2(z'_{f-3}-y'_{f-3})+...+p^f(z'_{f-1}-y'_{f-1})$ and we have:

$$\begin{aligned} m &\le \frac{p(z'_{f-2}-y'_{f-2})+p^2(z'_{f-3}-y'_{f-3})+...+p^f(z'_{f-1}-y'_{f-1})}{p^{f-1-k}} \\ &= (z'_k-y'_k)+p(z'_{k-1}-y'_{k-1})+...+p^{k+1}(z'_{f-1}-y'_{f-1})+\frac{(z'_{f-2}-y'_{f-2})}{p^{f-2-k}}+...+\frac{(z'_{k+1}-y'_{k+1})}{p} \\ &< (p^f-1)e+(z'_k-y'_k)+p(z'_{k-1}-y'_{k-1})+...+p^{f-1}(z'_{k+1}-y'_{k+1}) \\ &= \alpha(k,e)=m \end{aligned}$$

Contradiction. Therefore, $J^{AH}_{V_{\vec{t'},\vec{s'}}}(\chi_1,\chi_2)\neq J^{AH}_{V_{\vec{t''},\vec{s''}}}(\chi_1,\chi_2)$.

For Lemma 6.19(iii), we have:

$$y'_j = \begin{cases} 1 & \text{if } j=f-1 \\ 0 & \text{if } j\in T\smallsetminus\{f-1\} \end{cases} \tag{6.19.18}$$

$$y''_j = \begin{cases} 1 & \text{if } j=i \\ 0 & \text{if } j\in T\smallsetminus\{i\} \end{cases} \tag{6.19.19}$$

$$z'_j = \begin{cases} e-1+s'_j & \text{if } j=f-1 \\ e+s'_j & \text{if } j\in T\smallsetminus\{f-1\} \end{cases} \tag{6.19.20}$$

$$z''_j = \begin{cases} p+e-2+s'_j = p+(z'_j-y'_j) & \text{if } j=f-1 \\ p+e-1=(p-1)+(z'_j-y'_j) & \text{if } j\in[i+1,f-2] \\ e+s'_j=z'_j-y'_j & \text{if } j=i \\ e+s'_j=z'_j-y'_j & \text{if } j\notin[i,f-1] \end{cases} \tag{6.19.21}$$

$$\mathcal{I}'_j = \begin{cases} \{1\} \cup [s'_j+1, z'_j-1] & \text{if } j = f-1 \\ \{0\} \cup [s'_j+1, z'_j-1] & \text{if } j \in T \setminus \{f-1\} \end{cases} \tag{6.19.22}$$

$$\mathcal{I}''_j = \begin{cases} \{0\} \cup [p+s'_j-1, z''_j-1] & \text{if } j = f-1 \\ \{0\} \cup [p, z''_j-1] & \text{if } j \in [i+1, f-2] \\ \{1\} \cup [s'_j+2, z''_j-1] & \text{if } j = i \\ \{0\} \cup [s'_j+1, z'_j-1] & \text{if } j \notin [i, f-1] \end{cases} \tag{6.19.23}$$

$$V'_j = \begin{cases} [1, e-2] \cup \{z'_j - y'_j\} & \text{if } j = f-1 \\ [1, e-1] \cup \{z'_j - y'_j\} & \text{if } j \in T \setminus \{f-1\} \end{cases} \tag{6.19.24}$$

$$V''_j = \begin{cases} [1, e-1] \cup \{p + z'_j - y'_j\} & \text{if } j = f-1 \\ [2, e] \cup \{p + z'_j - y'_j\} & \text{if } j \in [i+1, f-2] \\ [2, e-1] \cup \{z'_j - y'_j\} & \text{if } j = i \\ [1, e-1] \cup \{z'_j - y'_j\} & \text{if } j \notin [i, f-1] \end{cases} \tag{6.19.25}$$

Every pair in P matches with some pair in P'' upon applying matching strategies in Remark 6.16(i) and Remark 6.16(ii). Therefore, $J^{AH}_{V_{\vec{t'},\vec{s'}}}(\chi_1, \chi_2) = J^{AH}_{V_{\vec{t''},\vec{s''}}}(\chi_1, \chi_2)$!

We omit demonstrating the calculations for Lemma 6.19(iv), Lemma 6.19(v) and Lemma 6.19(vi). They proceed similar to the calculations above, and the findings for all pairs described in Lemma 6.19 can be summarized as follows:

Proposition 6.20. *Suppose $V_{\vec{t'},\vec{s'}}$ and $V_{\vec{t''},\vec{s''}}$ are a pair of non-isomorphic, non-Steinberg Serre weights.*

Then there exist G_K characters χ_1 and χ_2 such that $|J^{AH}_{V_{\vec{t'},\vec{s'}}}(\chi_1, \chi_2)| = ef-1$ via Lemma 4.21(iii), $|J^{AH}_{V_{\vec{t''},\vec{s''}}}(\chi_1, \chi_2)| = ef-1$ via Lemma 4.21(iii) and $J^{AH}_{V_{\vec{t'},\vec{s'}}}(\chi_1, \chi_2) = J^{AH}_{V_{\vec{t''},\vec{s''}}}(\chi_1, \chi_2)$ if and only if

- *After reindexing if necessary, $\vec{s'}$ and $\vec{s''}$ satisfy either of the below for some $i < f-1$:*
  - $(s'_{f-1}, s'_{f-2}, ..., s'_{i+1}, s'_i) = (1, 0, ..., 0, \in [0, p-2])$;
    $(s''_{f-1}, s''_{f-2}, ..., s''_{i+1}, s''_i) = (p-1, p-1, ..., p-1, s'_i+1)$ *(Lemma 6.19(iii)).*
  - $s'_{f-1} \in [1, p-1]$, $(s'_i, s'_{i-1} ..., s'_0) = (p-1, p-1, ..., p-1)$;
    $s''_{f-1} = s'_{f-1} - 1$, $(s''_i, s''_{i-1} ..., s''_0) = (1, 0, ..., 0)$ *(Lemma 6.19(v)).*
- *With i as above, $\sum_{j\in T} d''_j \equiv 1 - p^{f-1-i} + \sum_{j \in T} t'_j$, and $\sum_{j\in T} t_j \equiv 1 + \sum_{j\in T} t'_j \mod p^f - 1$.*

Remark 6.21. In both the cases listed in Proposition 6.20, $(s_{f-1}, s_{f-2}, ..., s_0) \equiv (s'_{f-1} - 2, s'_{f-2}, ..., s'_0)$. We leave the precise specification of the highest weight to the reader.

6.21.1. *Case 3.* : $|J^{AH}_{V_{\vec{t'},\vec{s'}}}(\chi_1, \chi_2)| = ef-1$ via Lemma 4.21(i); $|J^{AH}_{V_{\vec{t''},\vec{s''}}}(\chi_1, \chi_2)| = ef-1$ via Lemma 4.21(iii).

Case 3a. : $i = f-1$.

Comparing ways of writing $\chi_2^{-1}\chi_1$ in terms of $\vec{s}$, $\vec{s'}$ and $\vec{s''}$, we have:

$$e-2-s'_{f-1}+\sum_{j\in T\smallsetminus\{f-1\}}p^{f-1-j}(s'_j+e)$$
$$\equiv (e-2+s''_{f-1})+\sum_{j\in T\smallsetminus\{f-1\}}p^{f-1-j}(s''_j+e)\equiv\sum_{j\in T}p^{f-1-j}(s_j+e)$$

(6.21.1)
$$\iff (-2-s'_{f-1},s'_{f-2},...,s'_0)\equiv(-2+s''_{f-1},s''_{f-2},...,s''_0)\equiv(s_{f-1},s_{f-2},...,s_0).$$

Comparing ways of writing χ_2, we have:

$$1+\sum_{j\in T}p^{f-1-j}d''_j\equiv s'_{f-1}+1\equiv\sum_{j\in T}p^{f-1-j}t_j \mod p^f-1$$
$$\iff \sum_{j\in T}p^{f-1-j}d''_j\equiv s'_{f-1},\quad \sum_{j\in T}p^{f-1-j}t_j\equiv s'_{f-1}+1 \mod p^f-1 \tag{6.21.2}$$

Lemma 6.22. *The condition in (6.21.1) is satisfied for some $\vec{s'}$, $\vec{s''}$ and $\vec{s}$ if and only if one of the following pairs describe $\vec{s'}$ and $\vec{s''}$:*

(i) $(s'_{f-1},s'_{f-2},...,s'_{k+1},s'_k)=(\in[1,p-2],0,...,0,\in[1,p-1])$ *where* $k\in[0,f-2]$*;*
$(s''_{f-1},s''_{f-2},...,s''_{k+1},s''_k)=(p-s'_{f-1},p-1,...,p-1,s'_k-1)$.

(ii) $(s'_{f-1},s'_{f-2},...,s'_0)=(\in[1,p-2],0,...,0)$*;*
$(s''_{f-1},s''_{f-2},...,s''_0)=(p-s'_{f-1}-1,p-1,...,p-1)$.

Proof. Easy verification upon recalling that $s'_{f-1}\leq p-2$ by Lemma 4.21(i) and $s''_{f-1}\geq 1$ by Lemma 4.21(iii). □

We omit the calculations for Lemma 6.22(i) which show that $J^{AH}_{V_{\vec{t'},\vec{s'}}}(\chi_1,\chi_2)\neq J^{AH}_{V_{\vec{t''},\vec{s''}}}(\chi_1,\chi_2)$. Briefly, $(f-1,e-1)\in P$ and $(k,e)\in P''$ are the pairs in P and P'' that don't match using matching strategies Remark 6.16(i) and Remark 6.16(ii). Both $\beta(f-1,e-1)$ and $\beta(k,e)$ turn out to have p-adic valuation 0 and therefore, by Remark 6.14, $\alpha(f-1,e-1)\neq\alpha(k,e)$.

For Lemma 6.22(ii), all pairs in P end up matching with some pair in P'' via Remark 6.16(i) or Remark 6.16(ii) (details omitted). Therefore, in this situation, $J^{AH}_{V_{\vec{t'},\vec{s'}}}(\chi_1,\chi_2)=J^{AH}_{V_{\vec{t''},\vec{s''}}}(\chi_1,\chi_2)$.

Case 3b. : $i < f-1$.

Comparing ways of writing $\chi_2^{-1}\chi_1$ in terms of $\vec{s'}$, $\vec{s''}$ and $\vec{s}$, we have:

$$e-2-s'_{f-1}+\sum_{j\in T\smallsetminus\{f-1\}}p^{f-1-j}(s'_j+e)$$
$$\equiv p^{f-1-i}(e-2+s''_i)+\sum_{j\in T\smallsetminus\{i\}}p^{f-1-j}(s''_j+e)\equiv\sum_{j\in T}p^{f-1-j}(s_j+e)$$

(6.22.1)
$$\iff (-2-s'_{f-1},s'_{f-2},...,s'_0)\equiv(s''_{f-1},...,s''_{i+1},-2+s''_i,s''_{i-1},...,s''_0)$$
$$\equiv(s_{f-1},s_{f-2},...,s_0)$$

Comparing ways of writing χ_2, we have:

$$p^{f-1-i}+\sum_{j\in T}p^{f-1-j}d''_j \equiv s'_{f-1}+1\equiv \sum_{j\in T}p^{f-1-j}t_j \mod p^f-1$$

(6.22.2)

$$\Longleftrightarrow \sum_{j\in T}p^{f-1-j}d''_j \equiv s'_{f-1}+1-p^{f-1-i},\quad \sum_{j\in T}p^{f-1-j}t_j \equiv s'_{f-1}+1 \mod p^f-1$$

Lemma 6.23. *The condition in (6.22.1) is satisfied for some $\vec{s'}$, $\vec{s''}$ and $\vec{s}$ if and only if one of the following pairs describe $\vec{s'}$ and $\vec{s''}$:*

(i) $(s'_{f-1},s'_{f-2},...,s'_{k+1},s'_k)=(\in[0,p-2],0,...,0,\in[1,p-1])$ *for some* $k\in[i+1,f-2]$,
$s'_i\in[0,p-3]$;
$(s''_{f-1},s''_{f-2},...,s''_{k+1},s''_k)=(p-2-s'_{f-1},p-1,...,p-1,s'_k-1)$,
$s''_i=s'_i+2$.

(ii) $(s'_{f-1},s'_{f-2},...,s'_{i+1},s'_i)=(\in[0,p-2],0,...,0,\in[0,p-2])$;
$(s''_{f-1},s''_{f-2},...,s''_{i+1},s''_i)=(p-2-s'_{f-1},p-1,...,p-1,s'_i+1)$.

(iii) $(s'_{f-1},s'_{f-2},...,s'_{k+1},s'_k)=(\in[0,p-2],0,...,0,\in[1,p-1])$ *for some* $k\in[i+1,f-2]$,
$(s'_i,s'_{i-1},...,s'_{l+1},s'_l)=(p-1,p-1,...,p-1,\in[0,p-2])$ *for some* $l\in[0,i-1]$;
$(s''_{f-1},s''_{f-2},...,s''_{k+1},s''_k)=(p-2-s'_{f-1},p-1,...,p-1,s'_k-1)$,
$(s''_i,s''_{i-1},...,s''_{l+1},s''_l)=(1,0,...,0,s'_l+1)$.

(iv) $(s'_{f-1},s'_{f-2},...,s'_{k+1},s'_k)=(\in[0,p-2],0,...,0,\in[1,p-1])$ *for some* $k\in[i+1,f-2]$,
$(s'_i,s'_{i-1},...,s'_0)=(p-1,p-1,...,p-1)$;
$(s''_{f-1},s''_{f-2},...,s''_{k+1},s''_k)=(p-1-s'_{f-1},p-1,...,p-1,s'_k-1)$,
$(s''_i,s''_{i-1},...,s''_0)=(1,0,...,0)$.

Proof. Easy verification upon recalling that $s'_{f-1}\le p-2$ by Lemma 4.21(i) and $s''_i\ge 1$ by Lemma 4.21(iii). ☐

For each of the pairs in the statement of Lemma 6.23, we omit the details of the calculations comparing $J^{AH}_{V_{\vec{t'},\vec{s'}}}(\chi_1,\chi_2)$ and $J^{AH}_{V_{\vec{t''},\vec{s''}}}(\chi_1,\chi_2)$. For pairs in Lemma 6.23 (i), (iii) and (iv), $J^{AH}_{V_{\vec{t'},\vec{s'}}}(\chi_1,\chi_2)\neq J^{AH}_{V_{\vec{t''},\vec{s''}}}(\chi_1,\chi_2)$. For the pair in Lemma 6.23(ii), $J^{AH}_{V_{\vec{t'},\vec{s'}}}(\chi_1,\chi_2)=J^{AH}_{V_{\vec{t''},\vec{s''}}}(\chi_1,\chi_2)$.

Proposition 6.24. *Suppose $V_{\vec{t'},\vec{s'}}$ and $V_{\vec{t''},\vec{s''}}$ are a pair of non-isomorphic, non-Steinberg Serre weights.*

Then there exist G_K characters χ_1 and χ_2 such that $|J^{AH}_{V_{\vec{t'},\vec{s'}}}(\chi_1,\chi_2)|=ef-1$ via Lemma 4.21(i), $|J^{AH}_{V_{\vec{t''},\vec{s''}}}(\chi_1,\chi_2)|=ef-1$ via Lemma 4.21(iii) and $J^{AH}_{V_{\vec{t'},\vec{s'}}}(\chi_1,\chi_2)=J^{AH}_{V_{\vec{t''},\vec{s''}}}(\chi_1,\chi_2)$ if and only if (after reindexing if necessary) $\vec{s'}$, $\vec{s''}$, $\vec{s}$, $\vec{t'}$ and $\vec{t''}$ and $\vec{t}$ satisfy either of the conditions below (we describe $\vec{s}$ only upto equivalence for the sake of clarity.):

- $(s'_{f-1},s'_{f-2},...,s'_0)=(\in[1,p-2],0,...,0)$;
$(s''_{f-1},s''_{f-2},...,s''_0)=(p-s'_{f-1}-1,p-1,...,p-1)$;
$(s_{f-1},s_{f-2},...,s_0)\equiv(p-3-s'_{f-1},p-1,...,p-1)$;
$\sum_{j\in T}p^{f-1-j}d''_j\equiv s'_{f-1}+\sum_{j\in T}p^{f-1-j}t'_j \mod p^f-1$, *and*

$\sum_{j\in T} p^{f-1-j}t_j \equiv s'_{f-1}+1+\sum_{j\in T} p^{f-1-j}t'_j \mod p^f-1$ *(Lemma 6.22(ii))*.

- *There exists some* $i \le f-2$ *such that:*
  $(s'_{f-1}, s'_{f-2}, ..., s'_{i+1}, s'_i) = (\in [0,p-2], 0, ..., 0, \in [0,p-2])$;
  $(s''_{f-1}, s''_{f-2}, ..., s''_{i+1}, s''_i) = (p-2-s'_{f-1}, p-1, ..., p-1, s'_i+1)$;
  $(s_{f-1}, s_{f-2}, ..., s_{i+1}, s_i, s_{i-1}, ..., s_0) \equiv (p-2-s'_{f-1}, p-1, ..., p-1, s'_i-1, s'_{i-1}, ..., s'_0)$;
  $\sum_{j\in T} p^{f-1-j}d''_j \equiv s'_{f-1}+1-p^{f-1-i}+\sum_{j\in T} p^{f-1-j}t'_j \mod p^f-1$, *and*
  $\sum_{j\in T} p^{f-1-j}t_j \equiv s'_{f-1}+1+\sum_{j\in T} p^{f-1-j}t'_j \mod p^f-1$ *(Lemma 6.23(ii))*.

Remark 6.25. For $p > 2$ and $i = f-2$, the second condition in the proposition above is identical to that required for $\mathrm{Ext}^1_{\mathbb{F}[\mathrm{GL}_2(k)]}(V_{\vec{t'},\vec{s'}}, V_{\vec{t''},\vec{s''}})$ to be non-zero via Proposition 2.1(i)(b). The relationship between $(\vec{t'},\vec{s'})$ and $(\vec{t''},\vec{s''})$ is asymmetric showing that only 1 family sees the type II intersection. For such a family witnessing a type II intersection, the two associated type I intersections correspond to $\mathrm{Ext}^1_{\mathbb{F}[\mathrm{GL}_2(k)]}(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}}) \neq 0$ via Proposition 2.1(i)(b) and $\mathrm{Hom}_{\mathrm{GL}_2(k)}(V_{\vec{t},\vec{s}}, H^1(\mathrm{G}_K, V_{\vec{t''},\vec{s''}})) \neq 0$. In particular, when $e > 1$, $f > 1$,
$\mathrm{Ext}^1_{\mathbb{F}[\mathrm{GL}_2(k)]}(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}}) \neq 0$ guarantees the existence of both a type I intersection and a type II intersection, while $\mathrm{Hom}_{\mathrm{GL}_2(k)}(V_{\vec{t},\vec{s}}, H^1(\mathrm{G}_K, V_{\vec{t''},\vec{s''}})) \neq 0$ only guarantees a type I intersection.

7. Conclusion

Theorem 7.1. *Let* $p > 2$ *be a fixed prime. Let* K *be a finite extension of* $\mathbb{Q}_p$*, with ring of integers* $\mathcal{O}_K$ *and residue field* k*. Set* $e = e(K/\mathbb{Q}_p)$*,* $f = f(K/\mathbb{Q}_p)$*. Let* $\mathcal{X}$ *be the reduced part of the Emerton-Gee stack for* GL_2 *constructed in* [EG1]*, defined over a finite field* $\mathbb{F}$*. Let* $V_{\vec{t},\vec{s}}$ *and* $V_{\vec{t'},\vec{s'}}$ *be a pair of non-isomorphic, non-Steinberg Serre weights for* $\mathrm{GL}_2(k)$*. Consider the irreducible component* $\mathcal{X}_{V_{\vec{t},\vec{s}}}$ *(resp.* $\mathcal{X}_{V_{\vec{t'},\vec{s'}}}$*) of* $\mathcal{X}$ *with the property that* $\overline{\rho} \in \mathcal{X}(\overline{\mathbb{F}})$ *is a point of* $\mathcal{X}_{V_{\vec{t},\vec{s}}}$ *(resp.* $\mathcal{X}_{V_{\vec{t'},\vec{s'}}}$*) if and only if* $V_{\vec{t},\vec{s}}$ *(resp.* $V_{\vec{t'},\vec{s'}}$*) is a Serre weight of* $\overline{\rho}$*.*

Then $\mathcal{X}_{V_{\vec{t},\vec{s}}}$ *and* $\mathcal{X}_{V_{\vec{t'},\vec{s'}}}$ *intersect in codimension* 1 *if and only if one of the following list of criteria holds. Next to each criterion we indicate in parenthesis the type of intersection.*

(i) $V_{\vec{t},\vec{s}}$ *and* $V_{\vec{t'},\vec{s'}}$ *are both weakly regular and* $\mathrm{Ext}^1_{\mathbb{F}[\mathrm{GL}_2(\mathcal{O}_K)]}(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}}) \neq 0$. *(Type I if* $e = 1$*, Type I or II or both if* $e > 1$*).*

(ii) $V_{\vec{t},\vec{s}}$ *and* $V_{\vec{t'},\vec{s'}}$ *are not both weakly regular,* $f > 1$ *and one of the following is true after possibly interchanging* $V_{\vec{t},\vec{s}}$ *and* $V_{\vec{t'},\vec{s'}}$ *and possibly changing the indices of* $\{s_j\}_j$*,* $\{s'_j\}_j$*,* $\{t_j\}_j$ *and* $\{t'_j\}_j$ *by adding a fixed integer. We also indicate when* $\mathrm{Ext}^1_{\mathbb{F}[\mathrm{GL}_2(k)]}(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}})$ *is non-vanishing, or when* $\mathrm{Ext}^1_{\mathbb{F}[\mathrm{GL}_2(k)]}(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}})$ *is vanishing but* $\mathrm{Ext}^1_{\mathbb{F}[\mathrm{GL}_2(\mathcal{O}_K)]}(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}})$ *is non-vanishing, or when* $\mathrm{Hom}_{\mathrm{GL}_2(k)}(V_{\vec{t},\vec{s}}, H^1(\mathcal{K}_1, V_{\vec{t'},\vec{s'}}))$ *is non-vanishing but it is not known whether it contributes to* $\mathrm{Ext}^1_{\mathbb{F}[\mathrm{GL}_2(\mathcal{O}_K)]}(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}})$ *or not. If nothing is mentioned, it means that* $\mathrm{Ext}^1_{\mathbb{F}[\mathrm{GL}_2(\mathcal{O}_K)]}(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}})$ *is vanishing.*
  (a) $(s_{f-1}, s_{f-2}, ..., s_{f-i}, s_{f-1-i}) = (\in [0,p-2], p-1, ..., p-1, \in [0,p-2])$, *where* $i \in [1, f-1]$;
  $(s'_{f-1}, s'_{f-2}, ..., s'_{f-i}, s'_{f-1-i}) = (p-s_{f-1}-2, 0, ..., 0, s_{f-1-i}+1)$;

$\sum_{j\in T} p^{f-1-j}t'_j \equiv -1 - s'_{f-1} + \sum_{j\in T} p^{f-1-j}t_j \mod p^f - 1$ *(Type I).*
When $i = 1$*, this implies* $\mathrm{Ext}^1_{\mathbb{F}[\mathrm{GL}_2(k)]}(V_{\vec{t},\vec{s}}, V_{\vec{t}',\vec{s}'}) \neq 0$.

(b) $(s_{f-1}, s_{f-2}, ..., s_0) = (\in [0, p-3], p-1, ..., p-1)$;
$(s'_{f-1}, s'_{f-2}, ..., s'_0) = (p-3-s_{f-1}, 0, ..., 0)$;
$\sum_{j\in T} p^{f-1-j}t'_j \equiv -1 - s'_{f-1} + \sum_{j\in T} p^{f-1-j}t_j \mod p^f - 1$ *(Type I).*

(c) $(s_{f-1}, s_{f-2}, s_{f-3}, ..., s_0) = (p-1, p-2, p-1, ..., p-1)$;
$(s'_{f-1}, s'_{f-2}, s'_{f-3}, ..., s'_0) = (p-2, 0, 0, ..., 0)$;
$\sum_{j\in T} p^{f-1-j}t'_j \equiv -1 - s'_{f-1} + \sum_{j\in T} p^{f-1-j}t_j \mod p^f - 1$ *(Type I).*
When $f = 2$*, this implies* $\mathrm{Ext}^1_{\mathbb{F}[\mathrm{GL}_2(k)]}(V_{\vec{t},\vec{s}}, V_{\vec{t}',\vec{s}'}) \neq 0$.

When $e = 1$, $f > 1$*, we additionally have:*

(d) $(s_{f-1}, s_{f-2}) = (p-1, \in [0, p-3])$;
$(s'_{f-1}, s'_{f-2}) = (p-1, s_{f-2}+2)$;
$\sum_{j\in T} p^{f-1-j}t'_j \equiv -p + \sum_{j\in T} p^{f-1-j}t_j \mod p^f - 1$ *(Type I).*
This implies $\mathrm{Ext}^1_{\mathbb{F}[\mathrm{GL}_2(k)]}(V_{\vec{t},\vec{s}}, V_{\vec{t}',\vec{s}'}) = 0$, $\mathrm{Hom}_{\mathrm{GL}_2(k)}(V_{\vec{t},\vec{s}}, H^1(\mathcal{K}_1, V_{\vec{t}',\vec{s}'})) \neq 0$.

(e) $f > 2$,
$(s_{f-1}, s_{f-2}, s_{f-3}, ..., s_{f-i}, s_{f-1-i}) = (p-1, p-1, p-1, ..., p-1, \in [0, p-2])$*, where* $i \in [2, f-1]$;
$(s'_{f-1}, s'_{f-2}, s'_{f-3} ..., s'_{f-i}, s'_{f-1-i}) = (p-1, 1, 0, ..., 0, s_{f-1-i}+1)$;
$\sum_{j\in T} p^{f-1-j}t'_j \equiv -1 - s'_{f-1} + \sum_{j\in T} p^{f-1-j}t_j \mod p^f - 1$ *(Type I).*

(f) $f = 2$,
$(s_{f-1}, s_{f-2}) = (p-2, p-1)$;
$(s'_{f-1}, s'_{f-2}) = (p-1, 1)$;
$\sum_{j\in T} p^{f-1-j}t'_j \equiv -1 - s'_{f-1} + \sum_{j\in T} p^{f-1-j}t_j \mod p^f - 1$ *(Type I).*

(g) $f > 2$,
$(s_{f-1}, s_{f-2}, s_{f-3}, ..., s_0) = (p-2, p-1, p-1, ..., p-1)$;
$(s'_{f-1}, s'_{f-2}, s'_{f-3}, ..., s'_0) = (p-1, 1, 0, ..., 0)$;
$\sum_{j\in T} p^{f-1-j}t'_j \equiv -1 - s'_{f-1} + \sum_{j\in T} p^{f-1-j}t_j \mod p^f - 1$ *(Type I).*

(h) $(s_{f-1}, s_{f-2}, ..., s_{f-1-i}, s_{f-2-i}) = (\in [0, p-2], 0, ..., 0, \in [1, p-2])$ *for some* $i \in [1, f-2]$;
$(s'_{f-1}, s'_{f-2}, ..., s'_{f-1-i}, s'_{f-2-i}) = (p - s_{f-1} - 2, p-1, ..., p-1, s_{f-2-i} + 1)$;
$s_{f-1} + 1 + \sum_{j=0}^{f-1} t_j \equiv p^{i+1} + \sum_{j=0}^{f-1} t'_j \mod p^f - 1$ *(Type II).*

(i) $f > 2$,
$(s_{f-1}, s_{f-2}, ..., s_{f-1-i}, s_{f-2-i}, s_{f-3-i}, ..., s_{f-m}, s_{f-1-m})$
$= (\in [0, p-2], 0, ..., 0, 0, 0, ..., 0, \in [1, p-1])$ *for some* $m \in [3, f-1]$;
$(s'_{f-1}, s'_{f-2}, ..., s'_{f-1-i}, s'_{f-2-i}, s'_{f-3-i}, ..., s'_{f-m}, s'_{f-1-m})$
$= (p - s_{f-1} - 2, p-1, ..., p-1, 1, 0, ..., 0, s_{f-1-m})$ *where* $i \in [1, m-2]$;
$s_{f-1} + 1 + \sum_{j=0}^{f-1} t_j \equiv p^i(s'_{f-1-i} + 1) + \sum_{j=0}^{f-1} t'_j \mod p^f - 1$ *(Type II).*

(j) $(s_{f-1}, s_{f-2}, ..., s_1, s_0) = (\in [0, p-3], 0, ..., 0, 0)$;
$(s'_{f-1}, s'_{f-2}, ..., s'_1, s'_0) = (p-1-s_{f-1}, p-1, ..., p-1, p-1)$;
$s_{f-1} + 1 + \sum_{j=0}^{f-1} t_j \equiv p^{f-1}(s'_0 + 1) + \sum_{j=0}^{f-1} t'_j \mod p^f - 1$ *(Type II).*

(k) $f > 2$,
$(s_{f-1}, s_{f-2}, ..., s_{f-1-i}, s_{f-2-i}, s_{f-3-i}, ..., s_0) =$

$(\in [0,p-3],0,...,0,0,0,...,0)$;
$(s'_{f-1},s'_{f-2},...,s'_{f-1-i},s'_{f-2-i},s'_{f-3-i},...,s'_0) =$
$(p-2-s_{f-1},p-1,...,p-1,1,0,...,0)$ *where* $i\in[1,f-2]$;
$s_{f-1}+1+\sum_{j=0}^{f-1}t_j \equiv p^i(s'_{f-1-i}+1)+\sum_{j=0}^{f-1}t'_j \mod p^f-1$ *(Type II).*

(l) $(s_{f-1},s_{f-2},...,s_{f-i},s_{f-1-i},s_{f-2-i},s_{f-3-i},...,s_0) =$
$(p-2,0,...,0,0,0,0,...,0)$;
$(s'_{f-1},s'_{f-2},...,s'_{f-i},s'_{f-1-i},s'_{f-2-i},s'_{f-3-i},...,s'_0) =$
$(0,p-1,...,p-1,p-1,1,0,...,0)$ *where* $i\in[2,f-1]$;
$s_{f-1}+1+\sum_{j=0}^{f-1}t_j \equiv p^i(s'_{f-1-i}+1)+\sum_{j=0}^{f-1}t'_j \mod p^f-1$ *(Type II).*

(m) $f=2$,
$(s_{f-1},s_{f-2})=(p-2,0)$;
$(s'_{f-1},s'_{f-2})=(1,p-1)$
$s_{f-1}+1+\sum_{j=0}^{f-1}t_j \equiv p^2+\sum_{j=0}^{f-1}t'_j \mod p^f-1$ *(Type II).*

(n) $f>2$,
$(s_{f-1},s_{f-2},s_{f-3},s_{f-4}...,s_0) =$
$(p-2,0,0,...,0)$;
$(s'_{f-1},s'_{f-2},s'_{f-3},...,s'_0)=(0,p-1,1,0,...,0)$;
$s_{f-1}+1+\sum_{j=0}^{f-1}t_j \equiv p(s'_{f-2}+1)+\sum_{j=0}^{f-1}t'_j \mod p^f-1$ *(Type II).*

(o) $f>2$,
$(s_{f-1},s_{f-2},s_{f-3},...,s_{f-1-i},s_{f-2-i}) = (p-1,1,0,...,0,\in[1,p-2])$ *where* $i>1$;
$(s'_{f-1},s'_{f-2},s'_{f-3},...,s'_{f-1-i},s'_{f-2-i}) = (p-1,p-1,p-1,...,p-1,s_{f-2-i}+1)$;
$s_{f-1}+1+\sum_{j=0}^{f-1}t_j \equiv p^i(s'_{f-1-i}+1)+\sum_{j=0}^{f-1}t'_j \mod p^f-1$ *(Type II).*

(p) $f>2$
$(s_{f-1},s_{f-2},s_{f-3},...,s_1,s_0)=(p-1,1,0,...,0,p-1)$;
$(s'_{f-1},s'_{f-2},s'_{f-3},...,s'_1,s'_0)=(1,0,0,...,0,p-1)$;
$s_{f-1}+1+\sum_{j=0}^{f-1}t_j \equiv p^{f-1}(s'_0+1)+\sum_{j=0}^{f-1}t'_j \mod p^f-1$ *(Type II).*

When $e>1$*, we additionally have:*

(q) $s_{f-1}\le p-3$;
$s'_{f-1}=s_{f-1}+2$;
$\sum_{j\in T}p^{f-1-j}t'_j \equiv -1+\sum_{j\in T}p^{f-1-j}t_j \mod p^f-1$ *(Type I).*
This implies $\mathrm{Ext}^1_{\mathbb{F}[\mathrm{GL}_2(k)]}(V_{\vec{t},\vec{s}},V_{\vec{t'},\vec{s'}})=0$*, but* $\mathrm{Ext}^1_{\mathbb{F}[\mathrm{GL}_2(\mathcal{O}_K)]}(V_{\vec{t},\vec{s}},V_{\vec{t'},\vec{s'}})\neq 0$.

(r) $(s_{f-1},s_{f-2},...,s_{f-i},s_{f-1-i})=(p-1,p-1,...,p-1,\in[0,p-2])$*, where* $i\ge 1$;
$(s'_{f-1},s'_{f-2},...,s'_{f-i},s'_{f-1-i})=(1,0,...,0,s_{f-i-1}+1)$;
$\sum_{j\in T}p^{f-1-j}t'_j \equiv -1+\sum_{j\in T}p^{f-1-j}t_j \mod p^f-1$ *(Type I).*

(s) $(s_{f-1},s_{f-2},...,s_0)=(p-2,p-1,...,p-1)$;
$(s'_{f-1},s'_{f-2},...,s'_0)=(1,0,...,0)$;
$\sum_{j\in T}p^{f-1-j}t'_j \equiv -1+\sum_{j\in T}p^{f-1-j}t_j \mod p^f-1$ *(Type I).*

(t) $(s_{f-1},s_{f-2},...,s_{i+1},s_i)=(1,0,...,0,\in[0,p-2])$*, where* $i<f-1$;
$(s'_{f-1},s'_{f-2},...,s'_{i+1},s'_i)=(p-1,p-1,...,p-1,s_i+1)$;
$\sum_{j\in T}t'_j \equiv 1-p^{f-1-i}+\sum_{j\in T}t_j \mod p^f-1$ *(Type II).*

(u) $s_{f-1} \in [1, p-1]$, $(s_i, s_{i-1}..., s_0) = (p-1, p-1, ..., p-1)$, *where* $i < f-1$; $s'_{f-1} = s_{f-1} - 1$, $(s'_i, s'_{i-1}..., s'_0) = (1, 0, ..., 0)$; $\sum_{j\in T} t'_j \equiv 1 - p^{f-1-i} + \sum_{j \in T} t_j \mod p^f - 1$ *(Type II).*

(v) $(s_{f-1}, s_{f-2}, ..., s_0) = (\in [1, p-2], 0, ..., 0)$; $(s'_{f-1}, s'_{f-2}, ..., s'_0) = (p - s_{f-1} - 1, p-1, ..., p-1)$; $\sum_{j\in T} p^{f-1-j} t'_j \equiv s_{f-1} + \sum_{j\in T} p^{f-1-j} t_j \mod p^{f-1}$ *(Type II).*

(w) $(s_{f-1}, s_{f-2}, ..., s_{i+1}, s_i) = (\in [0, p-2], 0, ..., 0, \in [0, p-2])$ *for some* $i \leq f-2$; $(s'_{f-1}, s'_{f-2}, ..., s'_{i+1}, s'_i) = (p-2-s_{f-1}, p-1, ..., p-1, s_i+1)$; $\sum_{j\in T} p^{f-1-j} t'_j \equiv s_{f-1} + 1 - p^{f-1-i} + \sum_{j\in T} p^{f-1-j} t_j \mod p^f - 1$ *(Type II). When* $i = f-2$, *this agrees with Item (a), corresponds to both Type I and II intersections, and implies* $\mathrm{Ext}^1_{\overline{\mathbb{F}}[\mathrm{GL}_2(k)]}(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}}) \neq 0$.

Proof. By Section 4.13, we need to find the criteria for when there exist two G_K characters χ_1 and χ_2 such that $L_{V_{\vec{t},\vec{s}}}(\chi_1, \chi_2) \cap L_{V_{\vec{t'},\vec{s'}}}(\chi_1, \chi_2) \subset \mathrm{Ext}^1_{\mathrm{G}_K}(\chi_2, \chi_1)$ has dimension $ef - 1$ and the same is true for most unramified twists of χ_1 and χ_2. The criteria are covered in Propositions 5.2, 5.6, 6.2, 6.4, 6.18, 6.20 and 6.24. In each of these, we have constraints on $\vec{s}$ and $\vec{s'}$ that do not depend on $\vec{t}$ and $\vec{t'}$. We similarly have constraints on $\sum_{j=0}^{f-1} p^{f-1-j} t'_j - \sum_{j=0}^{f-1} p^{f-1-j} t_j \mod p^f - 1$ that also do not depend on $\vec{t}$ and $\vec{t'}$. Collectively, the two sets of constraints define the criteria completely.

In other words, if and only if $V_{\vec{t},\vec{s}}$ and $V_{\vec{t'},\vec{s'}}$ satisfy the criteria in one of Propositions 5.2, 5.6, 6.2, 6.4, 6.18, 6.20 and 6.24.

The above criteria are necessary and sufficient when $K \neq \mathbb{Q}_p$. When $K = \mathbb{Q}_p$, $\mathcal{X}_{V_{\vec{t},\vec{s}}} \cap \mathcal{X}_{V_{\vec{t'},\vec{s'}}}$ is codimension 1 if and only if either the above criteria hold or the intersection contains an irreducible representation (by **??**). The criterion for existence of irreducible representations in $\mathcal{X}_{V_{\vec{t},\vec{s}}} \cap \mathcal{X}_{V_{\vec{t'},\vec{s'}}}$ is given in Lemma 4.14.

Putting all the criteria together gives the list in the statement of the Theorem, along with Proposition 2.14 and Corollary 2.10 on computations of extensions of Serre weights as $\mathrm{GL}_2(\mathcal{O}_K)$-modules.

Remark 7.2. In fact, our criteria show that when σ and τ are non-isomorphic, non-Steinberg Serre weights, then

$$\mathrm{Ext}^1_{\overline{\mathbb{F}}[\mathrm{GL}_2(\mathcal{O}_K)]}(\sigma, \tau) \neq 0 \implies \dim \mathcal{X}_\sigma \cap \mathcal{X}_\tau = [K : \mathbb{Q}_p] - 1.$$

This follows from the stronger statement in the proof of Proposition 2.14 when $K/\mathbb{Q}_p$ is unramified and Corollary 2.10 when $K/\mathbb{Q}_p$ is ramified.

□

Theorem 7.3. *In the setup of Theorem 7.1, assume that* $V_{\vec{t},\vec{s}}$ *and* $V_{\vec{t'},\vec{s'}}$ *are both weakly regular and that* $\mathcal{X}_{V_{\vec{t},\vec{s}}}$ *and* $\mathcal{X}_{V_{\vec{t'},\vec{s'}}}$ *intersect in codimension* 1. *Let* n *be the number of* $[K : \mathbb{Q}_p] - 1$ *dimensional irreducible components in the intersection. Then the following are true:*

(i) If $e = 1$, *then* $n = 1$.

None of these components of dimension $[K : \mathbb{Q}_p] - 1$ *are contained in triple intersections of irreducible components of* $\mathcal{X}$.

(ii) If $e > 1$ and $f = 1$, then

$$n = \begin{cases} 2 & \text{if } \mathrm{Ext}^1_{\mathbb{F}[\mathrm{GL}_2(k)]}(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}}) \neq 0 \\ 1 & \text{if } \mathrm{Ext}^1_{\mathbb{F}[\mathrm{GL}_2(k)]}(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}}) = 0 \end{cases}$$

If $s, s' < p-3$, then each component of dimension $[K : \mathbb{Q}_p] - 1$ is contained in a triple intersection.

(iii) If $e > 1$ and $f > 1$, then

$$n = \begin{cases} 2 & \text{if } \mathrm{Ext}^1_{\mathbb{F}[\mathrm{GL}_2(k)]}(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}}) \neq 0 \\ 1 & \text{if } \mathrm{Ext}^1_{\mathbb{F}[\mathrm{GL}_2(k)]}(V_{\vec{t},\vec{s}}, V_{\vec{t'},\vec{s'}}) = 0 \end{cases}$$

Each of these components of dimension $[K : \mathbb{Q}_p] - 1$ is contained in a triple intersection.

Proof. When $e = 1$, the statements are a consequence of collating criteria for type I and type II intersections in Corollary 5.7 and Proposition 6.4. When $f = 1$, the relevant results are in Propositions 5.2 and 6.2. When $e > 1$, $f > 1$, they are in Corollary 5.7 and Remark 6.25. □

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