GOAL : p-adic generalization of a modular form to eventually (not today) understand p-adic congruences of q-expansion coefficients of modular forms.

IDEA :

Let $n \geq 3$, $k \geq 2$ or $k=1$ & $n \in [3, 11]$, R a ring with n invertible.
Let $\overline{M}_n$ be the compactified modular curve for level n, defined over $\mathbb{Z}[\frac{1}{n}]$

$$R \otimes_{\mathbb{Z}[\frac{1}{n}]} \underbrace{H^0(\overline{M}_n, \underline{\omega}^{\otimes k})}_{\text{Modular forms of weight } k \text{ over } \mathbb{Z}[\frac{1}{n}]} \xrightarrow{\sim} H^0(\overline{M}_n, R \otimes \underline{\omega}^{\otimes k}) \overset{\text{Affine base change}}{\cong} \underbrace{H^0(\overline{M}_n \times \operatorname{Spec} R, \underline{\omega}^{\otimes k})}_{\text{Modular forms of wt } k \text{ over } \mathbb{Z}/p^r\mathbb{Z}}$$

Therefore, modular forms over $\mathbb{C}$ & those over $\mathbb{Z}/p^r\mathbb{Z}$ are obtained by base changing the same space. We don't get anything "extra" p-adically.

So instead of looking at all of $\overline{M}_n \otimes \mathbb{Z}_p$, we want to restrict to certain "rigid-analytic" open subsets of $\overline{M}_n \otimes \mathbb{Z}_p$ where certain classical modular forms become p-adically invertible.

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§1. HASSE INVARIANT

Let R be an $\mathbb{F}_p$-algebra
Let $E \xrightarrow{\pi} \operatorname{Spec} R$ be an elliptic curve.

By Serre duality $\underline{\omega} = \pi_* \Omega^1_{E/R} \xrightarrow{\sim} (R^1\pi_* \mathcal{O}_E)^\vee$

Suppose ω is a base of $\underline{\omega}$: $R\omega = H^0(E, \Omega^1_{E/R}) \xrightarrow{\sim} H^1(E, \mathcal{O}_E)^\vee$

Let $\eta \in H^1(E, \mathcal{O}_E)$ be such that η is dual to ω under the duality map above.
($\Rightarrow$ $\lambda^{-1}\eta$ is the dual of $\lambda\omega$ for $\lambda \in R^\times$)

Now, consider the absolute Frobenius map inducing a p-linear endomorphism F^*_{abs} on $\mathcal{O}_E$ $(x \longmapsto x^p)$ & therefore on $H^1(E, \mathcal{O}_E)$.

Let $A(E, \omega)$ be defined so that $F^*_{abs}(\eta) = A(E,\omega) \cdot \eta$

$$\boxed{A(E, \lambda\omega)}\, \lambda^{-1}\eta = F^*_{abs}(\lambda^{-1}\eta) \underset{\substack{\uparrow \\ \text{p-linear map}}}{=} \lambda^{-p} A(E,\omega)\,\eta = \boxed{\lambda^{1-p} A(E,\omega)}\, \lambda^{-1}\eta$$

So, $A(E,\omega)$ is a modular form of level one and weight $p-1$ defined over $\mathbb{F}_p$, called the <u>Hasse invariant</u>. It corresponds to the section $A(E,\omega)\,\omega^{\otimes p-1}$ of $\underline{\omega}^{\otimes p-1}$.

The Hasse invariant is holomorphic at ∞:

Consider $T(q) \xrightarrow{\pi} \mathrm{Spec}\, \mathbb{F}_p[[q]]$

$\exists$ a sheaf ω^0 on $T(q)$ s.t. ω^0 is invertible & the dualizing sheaf, s.t. $\pi_* \omega^0 \xrightarrow{\sim} R^1\pi_* \mathcal{O}_{T(q)/\mathbb{F}_p[[q]]}$, where the latter is invertible as before.

(Note: for smooth curves ω_0 is Ω^1, but here it is not)

ω_{can}, the canonical differential of $T(q)$ over $\mathbb{F}_p((q))$, is the restriction of a base of $\pi_*\omega^0$. Thus ω_{can} determines η_{can} of $H^1(T(q), \mathcal{O}_{T(q)})$ as $\mathbb{F}_p[[q]]$ module.

$A(T(q), \omega_{can})$ is the matrix of F_{abs}^* on $H^1(T(q), \mathcal{O}_{T(q)})$ w.r.t. the base η_{can}

$\Rightarrow A(T(q), \omega_{can}) \in \mathbb{F}_p[[q]]$

Its q-expansion turns out to be 1.

<u>Idea of pf</u>: For $E \underset{e}{\overset{\pi}{\rightleftarrows}} R$, R reduced, $e^*\Omega_1 \cong \pi_*\Omega_1$

e — identity section

$\therefore H^0(E, \Omega^1) =$ Cotangent space at the origin

$\| $ global sections of $\pi_*\Omega_1$

$\therefore H^1(E, \mathcal{O}_E) =$ tangent space at the origin

For a Tate curve, we take $D = t\partial t$, the invariant derivation dual to $\omega_{can} = \frac{dt}{t}$, the invariant differential. We consider the action of F_{abs} on the derivation and find that it is unchanged. $\therefore A(T(q), \omega_{can}) = 1$

$E_{p-1} \equiv A \mod p$

For $p \geq 5$, E_{p-1} is the modular form $\boxed{1 - \frac{2(p-1)}{b_{p-1}} \sum \sigma_{p-2}(n) q^n}$ of wt $p-1$,

Eisenstein

where $\sigma_{p-2}(n) = \sum_{\substack{d|n \\ d\geq 1}} d^{p-2}$. The q-expansion coefficients lie in $\mathbb{Q} \cap \mathbb{Z}_p$ as $v_p\left(\frac{-2(p-1)}{b_{p-1}}\right) = 1$

$\therefore A \equiv E_{p-1} \mod p.$

For $p = 2, 3$, not possible to lift A to a modular form of level 1, hol at ∞, over $\mathbb{Q} \cap \mathbb{Z}_p$ as spaces of the correct wt are 0-dim over $\mathbb{C}$.

> Theorem 1.7.1. Let $n \geq 3$, and suppose either that $k \geq 2$ or that $k = 1$ and $n \leq 11$. Then for any $\mathbb{Z}[1/n]$-module K, the canonical map
> $$K \otimes H^0(\overline{M}_n, \underline{\omega}^{\otimes k}) \longrightarrow H^0(\overline{M}_n, K \otimes (\underline{\omega}^{\otimes k}))$$
> is an isomorphism.

However, by base change formula,
for $p = 2$ ($\because$ wt $A = p-1 = 1$) and $n \in [3, 11]$, $2 \nmid n$ (s.t. $\mathbb{F}_2$ is a $\mathbb{Z}[\frac{1}{n}]$ module)
we can lift A to a modular form of level n and weight 1 over $\mathbb{Z}[\frac{1}{n}]$

for $p = 3$ ($\because$ wt $A = 2$) and any $n \geq 3$, $3 \nmid n$, we can lift A to a modular form of level n and weight 2

We fix such a lift and call it E_{p-1}

§2. P-ADIC MODULAR FORMS W/ GROWTH CONDITIONS

R_0 is a p-adically complete rg. Let $r \in R_0$. For $n \geq 1$, prime to p ($n \in [3, 11]$ for $p = 2$, and $n \geq 3$ for $p = 3$), define the module $\boxed{M(R_0, r, n, k)}$ of modular forms over R_0 of growth r, level n & wt k:

$f \in M(R_0, r, n, k)$ is a rule which assigns to any triple $(E/S, \alpha_n, Y)$ consisting of

(a) elliptic curve E/S, where S is an R_0-scheme on which p is nilpotent

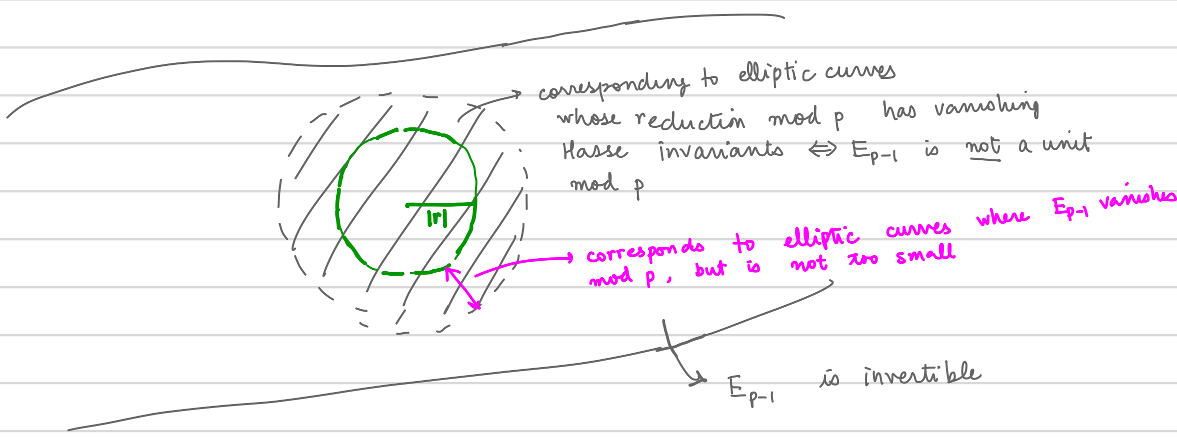
(b) level n structure α_n

(c) a section Y of $\omega^{\otimes(1-p)}$ satisfying $Y \cdot E_{p-1} = r$

Idea: Let $E_{p-1} = x\, \omega^{\otimes p-1}$, $x \in \mathcal{O}_S$
Let $Y = y\, \omega^{\otimes 1-p}$.

$\exists Y$: $Y E_{p-1} = xy = r$ $\iff$ $x \mid r$
necessarily unique if r is nzd
or "$1 \geq |E_{p-1}| \geq |r|$"

So for $r \in R_0^\times$, we are demanding that E_{p-1} be invertible $\iff$ reduction mod p $= A \neq 0$ $\iff$ E mod p is not supersingular
For a different r, we are "removing supersingular disks of radius $|r|$", whatever that means



a section $f(E/S, \alpha_n, Y)$ of $(\underline{\omega}_{E/S})^{\otimes k}$ over S, which depends only on the isom. class of the triple & which commutes w/ arbitrary base change of R_0-schemes.

Passage to the limit allows R ($S = \operatorname{Spec} R$) to not have a nilpotent p
(What we get on passage to the limit may not literally be a section of $\underline{\omega}_{E/S}$

f is holomorphic at ∞ if $\forall N \geq 1$, $f(T(q^n), \alpha_n, rE_{p-1}^{-1})$ considered over $\mathbb{Z}((q)) \otimes (R_0/p^N R_0)[\zeta_n]$ lies in $\mathbb{Z}[[q]] \otimes (R_0/p^N R_0)[\zeta_n]\, \omega_{can}^{\otimes k}$, for each level structure α_n

$S(R_0, r, n, k) \subset M(R_0, r, n, k)$ are those holomorphic at ∞.

By definitions,
$$M(R_0, r, n, k) = \varprojlim M(R_0/p^N R_0, r, n, k)$$
$$S(R_0, r, n, k) = \varprojlim S(R_0/p^N R_0, r, n, k)$$

Warning : Holomorphic forms, <u>NOT</u> cusp forms

§3, §4. DETERMINATION OF $M(R_0, r, n, k)$ & $S(R_0, r, n, k)$ WHEN p IS NILPOTENT IN R_0

Let's determine the universal triple $(E/S, \alpha_n, Y)$ for R_0, where p is nilpotent & $n \geq 3$ (s.t. M_n & $\overline{M}_n$ exist!). Let $\mathcal{L} = \underline{\omega}^{\otimes 1-p}$ (so Y should be a section of $\mathcal{L}$)

Consider the functor :

$$\mathcal{F}_{R_0, r, n} : S \longrightarrow \text{S-isom classes of triples } (E/S, \alpha_n, Y)$$
$$= \left\{ R_0\text{-morphisms } g : S \longrightarrow M_n \otimes R_0 \;+\; \text{a section } Y \text{ of } g^*\mathcal{L} \text{ satisfying } Y \cdot g^* E_{p-1} = r \right\}$$

picks out E/S, α_n

This is a subfunctor of the functor :

$$\mathcal{F}_{R_0, n} : S \longrightarrow \left\{ R_0\text{-morphisms } g : S \longrightarrow M_n \otimes R_0, \; + \text{ a section } Y \text{ of } g^*\mathcal{L} \right\}$$

$$= \left\{ R_0\text{-morphisms } g : S \longrightarrow M_n \otimes R_0, \; + \begin{array}{c} g^*\mathrm{Sym}(\check{\mathcal{L}}) \\ \cong \\ \mathrm{Sym}(g^*\check{\mathcal{L}}) \\ Y \uparrow \downarrow \\ S \end{array} \right\}$$

$$= \left\{ R_0\text{-morphisms } s : S \longrightarrow \underline{\mathrm{Spec}}_{M_n \otimes R_0}(\mathrm{Sym}(\check{\mathcal{L}})) \right\}$$

$$\begin{tikzcd}
S \arrow[d, dashed, "\text{determines } Y"'] \arrow[dr, bend left, "s"] \arrow[dd, bend right=60, "id"'] & \\
\mathrm{Spec}(\mathrm{Sym}\, g^*\check{\mathcal{L}}) \arrow[r] \arrow[d] & \underline{\mathrm{Spec}}_{M_n \otimes R_0}(\mathrm{Sym}(\check{\mathcal{L}})) \arrow[d, "t"] \\
S \arrow[r, "g = t \circ s"'] & M_n \otimes R_0
\end{tikzcd}$$

determines E/S, α_n

Now, $Y \in g^*\mathcal{L}$ corresponds to the map that sends x, a section of $g^*\check{\mathcal{L}}$, to $xY \in \mathcal{O}_S$
$\therefore$ $E_{p-1} \xrightarrow{Y} YE_{p-1} \in \mathcal{O}_S$. We want YE_{p-1} to be r $\Leftrightarrow$ $E_{p-1} - r \xrightarrow{Y} 0$

$\therefore$ $\mathcal{F}_{R_0,r,n}$ is represented by $V(E_{p-1}-r)$ in $\underline{\mathrm{Spec}}(\mathrm{Sym}(\check{\mathcal{L}}))$

Thus the universal triple $(E/S, \alpha_n, Y)$ is the inverse image on $\underline{\mathrm{Spec}}\,(\mathrm{Sym}\,\check{\mathcal{L}})$ of the universal elliptic curve w/ level n structure over $M_n \otimes R_0$.

reminder: p is nilpotent

$n \geq 3$

reminder: $\check{\mathcal{L}} = \omega^{\otimes p-1}$

$$M(R_0, r, n, k) = H^0(\underline{\mathrm{Spec}}_{M_n \otimes R_0}(\mathrm{Sym}\,\check{\mathcal{L}})/(E_{p-1}-r), \underline{\omega}^{\otimes k})$$

$$= H^0(M_n \otimes R_0, \underline{\omega}^{\otimes k} \otimes_{M_n \otimes R_0} \overbrace{(\bigoplus_{j\geq 0} \underline{\omega}^{\otimes j(p-1)})}^{\mathrm{Sym}\,\check{\mathcal{L}}}/E_{p-1}-r)$$

$$= H^0(M_n \otimes R_0, (\bigoplus_{j\geq 0} \underline{\omega}^{\otimes k+j(p-1)})/E_{p-1}-r)$$

$$= H^0(M_n \otimes R_0, \bigoplus_{j\geq 0} \underline{\omega}^{\otimes k+j(p-1)})/(E_{p-1}-r)$$

$M_n \otimes R_0$ is affine

$$= \bigoplus_{j \geq 0} M(R_0, n, k+j(p-1))/(E_{p-1}-r)$$

PROPOSITION: Let $n \geq 3$, $p \nmid n$. The submodule $S(R_0,r,n,k) \subset M(R_0,r,n,k)$ is the submodule $H^0(\underline{\mathrm{Spec}}_{\overline{M}_n \otimes R_0}\,\mathrm{Sym}\,\check{\mathcal{L}}/(E_{p-1}-r), \underline{\omega}^{\otimes k})$ of $H^0(\underline{\mathrm{Spec}}_{M_n \otimes R_0}\,\mathrm{Sym}\,\check{\mathcal{L}}/(E_{p-1}-r))$

Pf: To talk about q-expansions, we need to adjoin ζ_n (s.t. $T(q^n)$ has level structure), so assume $R_0 \ni \zeta_n$

The rg of completion of $\overline{M}_n \otimes R_0$ along ∞ is a finite number of copies of $R_0[[q]]$. Let one of the cusps be m_∞

$$(\overline{M}_n \otimes R_0)^\wedge_{m_\infty} = R_0[[q]]$$

Consider pullback of a cusp: $\underline{\mathrm{Spec}}_{\overline{M}_n \otimes R_0}\,\mathrm{Sym}\,\check{\mathcal{L}} \longrightarrow \overline{M}_n \otimes R_0$

freely gen by X ($= \omega_{can}^{\otimes p-1}$) on the local rg

$$(\overline{M}_n \otimes R_0)_{m_\infty} \otimes_{\overline{M}_n \otimes R_0} \frac{\mathrm{Sym}\,\check{\mathcal{L}}}{\underbrace{E_{p-1}}_{=aX} - r}$$

$$= \frac{(\overline{M}_n \otimes R_0)_{m_\infty}[X]}{aX - r}$$

Upon modding successively higher powers of m_∞, we know that E_{p-1} becomes invertible as $a \in R_0[[q]]^*$.

$\therefore$ already upon completing w.r.t. m_∞, we get $\dfrac{(M_n \otimes R_0)^\wedge_{m_\infty}[X]}{X = r/a} \cong R_0[[q]]$

$\therefore$ pullback of one cusp is exactly one cusp with completion of the stalk being $R_0[[q]]$.

$f \in H^0(\underline{\mathrm{Spec}}_{M_n \otimes R_0}(\mathrm{Sym}\,\check{\mathcal{L}})/(E_{p-1}-r), \underline{\omega}^{\otimes k})$ has holomorphic q-expansions if at the Tate curve, it $\in R_0[[q]]X \cong \underbrace{R_0[[q]]X \otimes \frac{\mathrm{Sym}\,\check{\mathcal{L}}}{E_{p-1}-r}}_{\text{completion at cusp of pullback}}$ $\iff f \in H^0(\underline{\mathrm{Spec}}_{\overline{M}_n \otimes R_0}\,\mathrm{Sym}\,\check{\mathcal{L}}/\ldots, \underline{\omega}^{\otimes 1-p})$

§5. DETERMINATION OF $S(R_0, r, n, k)$ IN THE LIMIT

Now, R_0 is any p-adically complete ring, $r \in R_0$ is not a zero divisor

We let $n \geq 3$ and :

we have a lift E_{p-1} of A for all these cases
- $k \geq 2$, or
- $k = 1$ and $n \leq 11$, or
- $k = 0$ and $p \neq 2$, or
- $k = 0$, $p = 2$, $n \leq 11$

All subsequent statements will apply to all the above cases. Proofs are sometimes different for different cases. We will only do the proofs for the general cases for the sake of clarity.

THEOREM : The homomorphism

$$\frac{\varprojlim H^0\left(\overline{M}_n, \bigoplus_{i \geq 0} \omega^{k+i(p-1)}\right) \otimes_{\mathbb{Z}[\frac{1}{n}]} R_0/p^N R_0}{E_{p-1} - r}$$

$$\downarrow$$

$$S(R_0, r, n, k) \overset{\text{definition}}{=} \varprojlim_N S(R_0/p^N R_0, r, n, k)$$

$$\overset{\text{shown already}}{\parallel}$$

$$\varprojlim_N H^0\left(\overline{M}_n, \frac{\bigoplus_{i \geq 0} \omega^{k+i(p-1)} \otimes R_0/p^N R_0}{E_{p-1} - r}\right)$$

is an isomorphism.

Pf : (We only do it for $k > 0$)

Let $\mathcal{S}$ be the quasicoherent sheaf $\bigoplus_{i \geq 0} \omega^{k+i(p-1)}$ on $\overline{M}_n$ & let $\mathcal{S}_N = \mathcal{S} \otimes R_0/p^N R_0$.

NOTE : As $k > 0$, base change for modular forms $\Rightarrow$

$$H^0(\overline{M}_n, \mathcal{S}) \otimes R_0/p^N R_0 \cong H^0(\overline{M}_n, \mathcal{S}_N)$$

So, we WTS that $\varprojlim H^0(\overline{M}_n, \mathcal{S}_N)/(E_{p-1} - r) \cong \varprojlim H^0(\overline{M}_n, \mathcal{S}_N/E_{p-1} - r)$

Consider the inverse system of exact sequences:

$$0 \longrightarrow \mathcal{S}_N \xrightarrow{E_{p-1}-r} \mathcal{S}_N \longrightarrow \mathcal{S}_N/(E_{p-1}-r) \longrightarrow 0$$

injective by degree considerations & the fact that r is n-z-d.

As $k>0$, $H^1(\overline{M}_n, \mathcal{S}_N) = 0$ (This is an argument in the proof of base change of modular forms in Ch 1).

$\therefore$ We get an SES of inverse systems:

$$\begin{array}{ccccccccc} 0 \to & H^0(\overline{M}_n, \mathcal{S}_N) & \xrightarrow{E_{p-1}-r} & H^0(\overline{M}_n, \mathcal{S}_N) & \longrightarrow & H^0(\overline{M}_n, \mathcal{S}_N/E_{p-1}-r) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & H^0(\overline{M}_n, \mathcal{S}_{N-1}) & \longrightarrow & H^0(\overline{M}_n, \mathcal{S}_N) & \longrightarrow & H^0(\overline{M}_n, \mathcal{S}_N/E_{p-1}-r) & \to 0 \end{array}$$

because of base change

Mittag-Leffler condition is satisfied & we get an SES of inverse limits as desired.

§6. DETERMINATION OF A BASIS OF $S(R_0, r, n, k)$ in the limit.

LEMMA: For each $j \geq 0$

(*) $$H^0(\overline{M}_n \otimes \mathbb{Z}_p, \underline{\omega}^{\otimes k+j(p-1)}) \xrightarrow{E_{p-1}} H^0(\overline{M}_n \otimes \mathbb{Z}_p, \underline{\omega}^{\otimes k+(j+1)(p-1)})$$

admits a section

Pf: Note, first, that E_{p-1} gives an injective map because $E_{p-1} \cdot x = 0 \Rightarrow E_{p-1}\, x = 0$ at the cusp $\Rightarrow$ q-expansion of x is 0 because E_{p-1} is invertible at the cusp $\Rightarrow x = 0$

Consider

$$0 \longrightarrow \underline{\omega}^{\otimes k+j(p-1)} \xrightarrow{E_{p-1}} \underline{\omega}^{\otimes k+(j+1)(p-1)} \longrightarrow \underline{\omega}^{k+(j+1)(p-1)}/E_{p-1} \longrightarrow 0$$

invertible sheaf

As $\overline{M}_n \otimes \mathbb{Z}_p$ is proper & flat over $\mathbb{Z}_p$, H^i of the sheaf is coherent & torsion free / $\mathbb{Z}_p$

$\therefore$ We get an exact sequence of finite free $\mathbb{Z}_p$-modules

$$0 \to H^0(\overline{M}_n \otimes \mathbb{Z}_p, \underline{\omega}^{\otimes k+j(p-1)}) \xrightarrow{E_{p-1}} H^0(\overline{M}_n \otimes \mathbb{Z}_p, \underline{\omega}^{k+(j+1)(p-1)}) \longrightarrow$$

$$\underbrace{H^0(\overline{M}_n \otimes \mathbb{Z}_p, \underline{\omega}^{\otimes k+(j+1)(p-1)}/E_{p-1}\underline{\omega}^{\otimes k+j(p-1)})}_{\text{is somehow } \mathbb{Z}_p\text{-flat by a theorem of Igusa, \& } \therefore \text{ is finite free}} \longrightarrow H^1(\overline{M}_n \otimes \mathbb{Z}_p, \underline{\omega}^{k+j(p-1)}) \longrightarrow 0$$

finite free $\leftarrow$ flatness $\Leftrightarrow$ torsion free

Grothendieck's coherence theorem

$H^1(\omega^{k+(j+1)(p-1)})$ vanishes for all our cases (argument in the proof of base change in Chapter 1)

$\because$ Cokernel of the map (*) is the kernel of a surjective map of finite free $\mathbb{Z}_p$-modules, $\therefore$ finite free, $\therefore$ we get splitting of

$$0 \longrightarrow H^0(\overline{M}_n \otimes \mathbb{Z}_p, \underline{\omega}^{\otimes k + j(p-1)}) \xrightarrow{E_{p-1}} H^0(\overline{M}_n \otimes \mathbb{Z}_p, \underline{\omega}^{k+(j+1)(p-1)}) \longrightarrow \text{coker} \longrightarrow 0$$

$\square$.

For each $j \geq 0$, fix a section of (*) and let the kernel of the section be $B(n,k,j+1) \subset$ weight $k+(j+1)(p-1)$ modular forms over $\mathbb{Z}_p$

i.e. we have for $j \geq 0$:

$$H^0(\overline{M}_n, \underline{\omega}^{\otimes k+(j+1)(p-1)}) \cong E_{p-1} H^0(\overline{M}_n, \underline{\omega}^{k+j(p-1)}) \oplus B(n,k,j+1)$$

$$\text{Let} \quad B(n,k,0) := H^0(\overline{M}_n, \underline{\omega}^{\otimes k})$$

Let $B(R_0, n, k, j) = B(n,k,j) \otimes_{\mathbb{Z}_p} R_0 \hookrightarrow H^0(\overline{M}_n, \underline{\omega}^{k+j(p-1)}) \otimes_{\mathbb{Z}_p} R_0$

no dependence on r, actually!!

Let $B^{\text{rigid}}(R_0, r, n, k)$ denote the R_0 module containing all formal sums $\sum_{a=0}^{\infty} b_a$, $b_a \in B(R_0, n, k, a)$ whose terms tend to 0 p-adically

($\forall N$, $\exists M$, s.t. $b_a \in p^N B(R, n, k, a)$ $\forall a \geq M$)

PROPOSITION :

$$\begin{array}{ccc}
\sum_j b_a \in B^{\text{rigid}}(R_0, r, n, k) & \hookrightarrow & \varprojlim_N H^0(\overline{M}_n, \bigoplus_{j\geq 0} \underline{\omega}^{k+j(p-1)}) \otimes_{\mathbb{Z}_p} R_0/p^N R_0 \\
& & \downarrow \\
& & \dfrac{\varprojlim_N H^0(\overline{M}_n, \bigoplus \underline{\omega}^{k+j(p-1)}) \otimes_{\mathbb{Z}_p} R_0/p^N R_0}{E_{p-1} - r} \\
& & \Big\downarrow \wr \leftarrow \text{proven already} \\
& & \sum_j b_a : ((E/S, \alpha_n, Y) \longmapsto \sum_j b_a(E/S, \alpha_n) Y_a) \in S(R_0, r, n, k)
\end{array}$$

The dashed arrow is an isomorphism

Pf : Injectivity :

Suppose $\sum_{a \geq 0} b_a \in B^{\text{rigid}}(R_0, r, n, k)$ can be written as $(E_{p-1} - r) \cdot \sum_{a \geq 0} s_a$ w/ $s_a \in S(R, n, k+a(p-1))$ & s_a tending to 0 as $a \to \infty$.

$b_a = 0$ iff $\forall N > 0$ $b_a \equiv 0 \mod p^N$ (Krull intersection theorem)

Mod p^N, $\sum_j b_a$ & $\sum_j s_a$ are finite sums. Suppose $b_a \equiv s_a \equiv 0 \mod p^N$ $\forall a > M$. As $0 \equiv b_{M+1} \equiv E_{p-1} s_M - r s_{M+1} \equiv E_{p-1} s_M$, we get $s_M \equiv 0$ (since E_{p-1} is nzd.)

$$b_M \equiv E_{p-1} s_{M-1} - r s_M \equiv E_{p-1} s_{M-1}$$

$$\underbrace{\overset{\cap}{B(n,k,M)} \qquad E_{p-1} H^0(\overline{M}_n, \underline{\omega}^{k+(M-1)(p-1)})}_{\text{trivial intersection}}$$

$$\therefore b_M = 0$$

Continuing $\sum b_a = 0 \mod p^N$

Surjectivity :

Notice that

$$S(R_0, n, k+j(p-1)) = H^0(\overline{M}_n, \underline{\omega}^{\otimes k+j(p-1)}) \xleftarrow{\sim} \bigoplus_{j=0}^{j} B(R_0, n, k, a)$$

$$\sum E_{p-1}^{j-a} b_a \longleftarrow \sum b_a$$

Given $\sum s_a$, $s_a \in S(R_0, n, k+a(p-1))$ tending to 0, we may decompose $s_a = \sum_{i+j=a} (E_{p-1})^i b_j(a)$ with $\underbrace{b_j(a)}_{\in B(R_0,n,k,j)}$ tending to 0 as $a \to \infty$ uniformly in j

Then $\sum_a s_a = \sum_a \sum_{i+j=a} (E_{p-1})^i b_j(a) = \sum_a \sum_{i+j=a} r^i b_j(a)$ in $S(R_0, r, n, k)$

$s(a) \to 0$ as $a \to \infty \Rightarrow b_j(a) \to 0$ as $a \to \infty$

For each j, $\sum_i r^i b_j(i+j)$ converges to $b_j' \in B(R_0, n, k, j)$ & $b_j' \to 0$ as $j \to \infty$ (as $b_j(a) \to 0$ uniformly in j)

$\therefore$ $\sum_{j \geq 0} b_j'$ exists & has same image in $S(R_0, r, n, k)$ as $\sum_{a \geq 0} s_a$

COROLLARY : The following is an injection :

$$S(R_0, r, n, k) \longrightarrow S(R_0, 1, n, k)$$

$$f \longmapsto \left((E/S, \alpha_n, Y) \longmapsto (E/S, \alpha_n, rY) \xmapsto{f} f(E/S, \alpha_n, rY) \right)$$

Pf :

$$\underset{B^{rigid}(R_0, r, n, k)}{\overset{\cap}{\sum b_a}} \longmapsto \left((E/S, \alpha_n, Y) \longmapsto \sum b_a(E/S, \alpha_n)(rY)^a = \sum r^a b_a(E/S, \alpha_n) Y^a \right)$$

$$\sum b_a(E/S, \alpha_n)(rY)^a = \sum r^a b_a \in B^{rigid}(R_0, 1, n, k)$$

$$\sum r^a b_a = 0 \Rightarrow r^a b_a = 0 \ \forall a \underset{r \text{ is nzd}}{\Rightarrow} b_a = 0 \ \forall a$$

□.

INTERPRETATION VIA FORMAL SCHEMES:

For R_0 p-adically complete and $r \in R_0$

completion along p of $\mathrm{Spec}_{M_n} \frac{\mathrm{Sym}\, \check{\mathcal{L}}}{E_{p-1} - r}$

Consider the formal scheme $M_n(R_0, r)$ corresponding to the functor

$$S \longmapsto \varinjlim_N \mathrm{Spec}_{M_n \otimes R_0/p^N R_0} (\mathrm{Sym}\, \check{\mathcal{L}} / (E_{p-1} - r))\ (S)$$

As M_n is affine, this is just the space $\mathcal{X}$ consisting of prime ideals of $\mathrm{Sym}\, \check{\mathcal{L}}/(E_{p-1}-r)$ that contain p with $\mathcal{O}_{\mathcal{X}} = \varprojlim_N \left(\mathcal{O}_{\mathrm{Sym}\, \check{\mathcal{L}}/..} / p^N\right)$

$\omega^{\otimes k}$ corresponds to a module F on $\mathcal{O}_{\mathrm{Sym}\, \check{\mathcal{L}}/(E_{p-1}-r)}$. $\forall N$, F/p^N gives us a mod p^N module, which gives us a quasicoherent sheaf on $\mathcal{X}$ whose global sections are

$$\varprojlim_N H^0\Big(M_n \otimes R_0/p^N R_0, \bigoplus_{j \geq 0} \omega^{\otimes k + j(p-1)} / E_{p-1} - r\Big) = M_n(R_0, r, n, k)$$

Similar stuff can be said for $\overline{M}_n$.

§7. q-expansion for $r = 1$

PROPOSITION: let $x \in R_0$ be s.t. $x | p^N$ for some $N \geq 1$. TFAE for $f \in S(R_0, 1, n, k)$:

(1) $f \in x S(R_0, 1, n, k)$

(2) q expansions of f all lie in $x \cdot R_0[\zeta_n][[q]]$

(3) On each of the $\varphi(n)$ connected components of $\overline{M}_n \otimes_{\mathbb{Z}[\frac{1}{n}]} \mathbb{Z}[\frac{1}{n}, \zeta_n]$, $\exists$ at least one cusp where the q-expansion of f lies in $x \cdot R_0[\zeta_n][[q]]$

Pf: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is clear.

We have

$$S(R_0/xR_0, 1, n, k) \cong B^{\mathrm{rigid}}(R_0/xR_0, 1, n, k) \xleftarrow{\sim} B^{\mathrm{rigid}}(R_0, 1, n, k)/x \cdot B^{\mathrm{rigid}}(R_0, 1, n, k)$$

Replacing R_0 by R_0/xR_0, we have $x = 0$ & p is nilpotent.

$\therefore f \in B^{\mathrm{rigid}}(R_0, 1, n, k)$ is a finite sum $\sum_{a=0}^{M} b_a$ & its q-expansion at $(T(q^n), \alpha_n, E_{p-1}^{-1})$ is that of

$$\sum_{a=0}^{M} b_a E_{p-1}^{-a} = \frac{\overbrace{\sum_{a=0}^{M} b_a E_{p-1}^{M-a}}^{\text{a true modular form by the way!}}}{E_{p-1}^{M}}$$

By hypothesis, $\sum_{a=0}^{M} b_a (E_{p-1})^{M-a}$ has q-expansion 0 at one or more cusps on each geometric connected component of $\bar{M}_n$. By q-expansion principle, $\sum b_a (E_{p-1})^{M-a} = 0$. By virtue of the isomorphism below, $\sum b_a = 0$.

$$\bigoplus_{a=0}^{M} B(R_0, n, k, a) \xrightarrow{\sim} S(R_0, n, k + M(p-1)) \qquad \text{(discussed earlier)}$$
$$\sum b_a \longmapsto \sum b_a E_{p-1}^{M-a}$$

□.

Cor: f has 0 q-expansion $\Rightarrow f = 0$. By (3) $\Rightarrow$ (1), $f \in p^N S(R_0, 1, n, k)$ $\forall N$, $\therefore$ is 0.

PROPOSITION: Suppose $\exists$ a power series $f_\alpha(q) \in R_0[\zeta_n][[q]]$ for each cusp α of $\bar{M}_n$. TFAE:

1) The f_α are q-expansions of an (necessarily unique) element $f \in S(R_0, 1, n, k)$

2) For every power p^N of p, $\exists$ $M \geq 1$ s.t. $M \equiv 0 \mod p^{N-1}$ and a "true" modular form $g_N \in S(R_0, n, k + M(p-1))$ whose q-expansions are congruent mod p^N to the given f_α.

Pf: (1) $\Rightarrow$ (2):

If g_N exists mod p^N, then we can lift it to a modular form in R_0. (Obvious by base change for $k>0$ conditions, but also true for the $k=0$ conditions specified earlier).

So, replace R_0 by $R_0/p^N R_0$ & suppose p is nilpotent.

WTS that f is the q-expansion of a true modular form of level n & wt $k' \geq k$, $k' \equiv k \mod p^{N-1}(p-1)$

As seen in proof of proposition above, for p nilpotent in R_0, f has the same q-expansions as g/E_{p-1}^{M} where $M >> 0$ & g is a true modular form of weight $k + M(p-1)$. Multiplying top & bottom by suitable power of E_{p-1}, WMA $M \equiv 0 \mod p^{N-1}$.

$E_{p-1}(q) \equiv 1 \mod p$ at each cusp $\Rightarrow$ $E_{p-1}^{p^{N-1}}(q) \equiv 1 \mod p^N$ & $\therefore E_{p-1}^{M}(q) \equiv 1 \mod p^N$

$\Rightarrow$ f mod p^N has same q-expansion as g.

(2) $\Rightarrow$ (1). Multiply g_N by powers of $E_{p-1}^{p^{N-1}}$ if needed s.t. WMA that weights $k + M_N(p-1)$ of the g_N are increasing with N

$g_{N+1} - g_N E_{p-1}^{M_{N+1} - M_N} \in p^N S(R_0, n, k + M_{N+1}(p-1))$ by q-expansion principle.

Take $g_0 = 0$

Hence. $\sum_N \left(g_{N+1} - g_N E_{p-1}^{M_{N+1}-M_N}\right)$ gives an element of $S(R_0, 1, n, k)$ whose q-expansions are congruent to those of g_N mod p^N.

§8. BASES FOR LEVELS 1 & 2

All of the above discussion needed $n \geq 3$, so that M_n was defined.

Suppose $p \neq 2, 3$. Then E_{p-1} is a modular form of level 1 lifting the Hasse inv.

For $n \geq 3$, prime to p, R_0 p-adically complete, $r \in R_0$, $GL_2(\mathbb{Z}/n\mathbb{Z})$ acts on the functor $\mathcal{F}_{R_0, r, n}$ by

$$g(E/S, \alpha_n, Y) = (E/S, g \circ \alpha_n, Y)$$

(as E_{p-1} doesn't depend on level, $E_{p-1}Y$ remains equal to r upon changing level)

This induces action on $M(R_0, r, n, k)$ and on $S(R_0, r, n, k)$.

Notice that
$$M(R_0, r, 1, k) = M(R_0, r, n, k)^{GL_2(\mathbb{Z}/n\mathbb{Z})}$$
&
$$S(R_0, r, 1, k) = S(R_0, r, n, k)^{GL_2(\mathbb{Z}/n\mathbb{Z})}$$

Now suppose $n = 3$ or $n = 4$. Then $GL_2(\mathbb{Z}/n\mathbb{Z})$ has order prime to $p \neq 2, 3$
$(|GL_2(\mathbb{Z}/3\mathbb{Z})| = 48, \quad |GL_2(\mathbb{Z}/4\mathbb{Z})| = 96)$

Consider the map $P = \dfrac{1}{\# GL_2(\mathbb{Z}/n\mathbb{Z})} \sum g$, giving a projection onto invariants

Define $B(1, k, j) = B(n, k, j)^{GL_2(\mathbb{Z}/n\mathbb{Z})} = P(B(n, k, j))$

$$B(R_0, 1, k, j) = B(R_0, n, k, j)^{GL_2(\mathbb{Z}/n\mathbb{Z})} = B(1, k, j) \otimes_{\mathbb{Z}[1/n]} R_0$$

↑ P has a section, being a projection. So commutes with base change

Define $B^{\text{rigid}}(R_0, r, 1, k) = P(B^{\text{rigid}}(R_0, r, n, k)) = (B^{\text{rigid}}(R_0, r, n, k))^{GL_2(\mathbb{Z}/n\mathbb{Z})}$
consisting of $\sum b_a$ where each b_a is invariant by $GL_2(\mathbb{Z}/n\mathbb{Z})$

Applying P to previous isomorphisms gives for r not a zero divisor

$$B^{\text{rigid}}(R_0, r, 1, k) \xrightarrow{\sim} S(R_0, r, 1, k)$$
$$\sum b_a \longmapsto \left((E/S, \alpha_1, Y) \mapsto \sum b_a(E/S, \alpha_1) Y^a\right)$$

Now, let $p \neq 2$ & consider level 2. Let $E_{p-1} \in S(\mathbb{Z}[\frac{1}{2}], 2, p-1)$ be a lifting of the Hasse Invariant.

Let G_1 = kernel : $GL_2(\mathbb{Z}/4\mathbb{Z}) \longrightarrow GL_2(\mathbb{Z}/2\mathbb{Z})$

Level 4 structure induces a level 2 structure as $E[2] \hookrightarrow E[4] \xrightarrow[\sim]{\alpha_4} (\mathbb{Z}/4\mathbb{Z})^2$. $g \in GL_2(\mathbb{Z}/4\mathbb{Z})$ leaves the level 2 structure unchanged iff $g \in G_1$

$\therefore$ G_1 invariants of level 4 modular forms give level 2 modular forms & the projector $P_1 = \frac{1}{\# G_1} \Sigma g_1$ gives us all the G_1 invariants.

Similar considerations as above give :

$$B^{\text{rigid}}(R_0, r, 2, k) \xrightarrow{\sim} S(R_0, r, 2, k) \quad \text{for } r \text{ not a zero divisor}$$