GOAL : Explain certain p-adic congruences using the theory of p-adic modular forms

OUTLINE

- Show that p-adic representations of the fundamental groups of certain schemes on which p is nilpotent correspond to certain coherent sheaves with a Frobenius action
- Representations corresponding to $\underline{\omega}$ on the ordinary locus are those coming from the étale quotient of $\ker[p^m]$.
- These representations are highly nontrivial. Triviality for k tensor powers gives congruence relations on values of k
- Definition of and application to modular forms of weight χ, $\chi \in \mathrm{End}(\mathbb{Z}_p^*)$

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§1. P-ADIC REPRESENTATIONS & LOCALLY FREE SHEAVES

Let q be a power of p, k a perfect field containing $\mathbb{F}_q$
$W_n(k)$: ring of Witt vectors of length n

S_n : flat, affine $W_n(k)$ scheme with normal, reduced, irreducible special fiber

Suppose S_n admits an endomorphism φ which induces the q power mapping on the special fibre.

Proposition:

$$\left\{\begin{matrix}\text{Finite free } W_n(\mathbb{F}_q) \text{ modules } M \\ \text{with continuous } \pi_1(S_n) \text{ action}\end{matrix}\right\} \underset{\text{equivalence}}{\overset{\text{cat.}}{\longleftrightarrow}} \left\{\begin{matrix}\text{Pairs } (H, F) \text{, where } H \text{ is locally} \\ \text{free sheaf of finite rank on } S_n \\ \text{\& } F \text{ is an isom } \varphi^*H \xrightarrow{\sim} H\end{matrix}\right\}$$

Map from left to right is given as follows:

Consider
$$\begin{tikzcd} \pi^1(S_n) \arrow[r] \arrow[dr, two heads] & \mathrm{Aut}_{W_n(\mathbb{F}_q)} M \\ & \mathrm{Aut}(T_n/S_n) \arrow[u, dashed] \end{tikzcd}$$

finite étale galois S_n-scheme

$\exists!$ φ-linear map φ_T on T_n inducing q-power endomorphism on special fibre

$$\begin{tikzcd} \varphi^*(T_n \times k) \arrow[r, hook] \arrow[d, hook] & T_n \times k \arrow[r, hook] & T_n \arrow[d, "\text{étale}"] \\ \varphi^* T_n \arrow[r] \arrow[urr, dashed, "\exists!"] & \varphi^* S_n \arrow[r] & S_n \end{tikzcd}$$

Let $H_T := M \otimes_{W_n(\mathbb{F}_q)} \mathcal{O}_{T_n}$

$F_T = id \otimes \varphi_T$

$g \in \mathrm{Aut}(T_n/S_n)$ acts on H_T via $m \otimes x \longmapsto g(m) \otimes g^{-1\#}x$ & commutes w/ φ_T

(H, F) is given by standard descent

uses flatness

F_T fixed points of $\mathcal{O}_{T_n}$ are just $W_n(\mathbb{F}_q)$
& M is recovered as the fixed points of F_T on global sections of H_T
(This gives fully faithful)

Essential surjectivity :

Given (H, F), WTS $\exists$ a finite étale cover T_n of S_n over which H admits a basis of F-fixed points.

Skipping the proof: Essentially for $n=1$, we reduce to the fact that a f.d. v.s. over $\bar{k}$ with a q-linear automorphism is spanned by its fixed points
↳ Hilbert's 90

For $n>1$, we do an induction argument by solving for the equations that guarantee that an F-fixed basis over S_{n-1} lifts to an F-fixed basis over S_n.

Remark 1: The categorical equivalence respects tensor products

Remark 2: étale site is "topologically invariant", so $\pi_1(S_m) = \pi_1(S_1)$

As S_1 is normal, reduced & irreducible, a representation of $\pi_1(S_1)$ is just a suitably unramified representation of the Galois gp of the function field of S_1

Let η be the generic pt of S_1

$$\begin{tikzcd} \mathrm{Gal}(\overline{k(\eta)}/k(\eta)) \arrow[r, "\sim"] & \pi_1(k(\eta), \eta) \arrow[d, two heads] \\ & \pi_1(S_1, \eta) \end{tikzcd}$$

Therefore, for a non-empty open $U \subset S_n$,

$$\{\text{Representations of } \pi_1(S_n)\} \xrightarrow{\text{restriction}} \{\text{Representations of } \pi_1(U)\}$$

is fully faithful

§2. APPLICATION TO MODULAR SCHEMES

Let $n \geq 3$, $p \nmid n$, q s.t. $W(\mathbb{F}_q) \supset$ primitive n'th roots of unity.

$$M_n \otimes W_m(\mathbb{F}_q) = \coprod_{\substack{\text{primitive} \\ n^{th} \text{ root}}} \text{smooth curves corresponding to e.m. pairing given by } \zeta$$

(Similar for $\overline{M}_n$)

Recall: p-adic modular forms with growth condition given by $r=1$ correspond to sections of $\underline{\omega}^{\otimes k}$ on

$\underline{\omega}^{\otimes p-1}$ on $M_n \otimes W_m(\mathbb{F}_q)$

$$M_n(W_m(\mathbb{F}_q), 1) = \mathrm{Spec}_{M_n \otimes W_m(\mathbb{F}_q)} \frac{\mathrm{Sym}\, \check{\mathcal{L}}}{E_{p-1} - 1} \cong D(E_{p-1}) \subset M_n \otimes W_m(\mathbb{F}_q)$$

as p is nilpotent, the formal scheme on LHS is an actual scheme

$$= \bigcup_{\substack{\zeta \text{ primitive} \\ n^{th} \text{ root}}} D(E_{p-1}) \cap \zeta\text{-component} = \bigcup S_m^{\zeta}$$

Similarly,

$$\overline{M}_n(W_m(\mathbb{F}_q), 1) = \cup\, \overline{S}_m^{\varsigma}$$

Note: S_m^{ς}, $\overline{S}_m^{\varsigma}$ are smooth affine $W_m(\mathbb{F}_q)$ schemes with geometrically connected fibres

Recall the Frobenius aut of $M(W_m(\mathbb{F}_q), 1, n, k)$ (meromorphic modular forms; growth condition) & of $S(W_m(\mathbb{F}_q), 1, n, k)$ (holomorphic modular forms):

$$(\varphi\cdot f)(E, \alpha_n, \underset{=E_{p-1}^{-1}}{Y}) = f(\underset{\text{canonical } p\text{-subgp}}{E/H}, \underset{\pi: E\to E/H}{\pi(\alpha_n)}, \underset{E_{p-1}^{-1} \text{ here as } r=1}{Y'})$$

We saw that φ gives a ring homomorphism on wt 0 forms, inducing an automorphism of the affine schemes $M(W_m(\mathbb{F}_q), 1)$ & $\overline{M}(W_m(\mathbb{F}_q), 1)$.

φ carries idempotents to idempotents & mod p, $\varphi(e_{\varsigma^p})(E, \alpha^n, Y) = e_{\varsigma^p}(E^{(\sigma)}, \pi(\alpha^n), Y')$

$$= \begin{cases} 1 & \text{if } E \in S^{\varsigma} \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow$ φ maps S_m^{ς} to $S_m^{\varsigma^p}$ & $\overline{S}_m^{\varsigma}$ to $\overline{S}_m^{\varsigma^p}$

But notice that for σ, the Frobenius aut of $W_m(\mathbb{F}_q)$, maps ς to ς^p.

Therefore $\varphi: S_m^{\varsigma} \longrightarrow S_m^{\varsigma^p} \cong S_m^{\varsigma} \times_{W_m(\mathbb{F}_q),\sigma} W_m(\mathbb{F}_q) = (S_m^{\varsigma})^{(\sigma)}$

We view φ as a σ-linear endomorphism of S_m^{ς}.
Similarly for $\overline{S}_m^{\varsigma}$

φ on $M(W_m(\mathbb{F}_q), 1, n, k)$ can be viewed as a φ-linear endomorphism of $\underline{\omega}^{\otimes k}|_{\overline{S}_m^{\varsigma}}$ for each primitive n^{th} root of unity ς.

Q: Which rep of $\pi_1(\overline{S}_m^{\varsigma})$ in a free $\mathbb{Z}/p^m\mathbb{Z} = W_m(\mathbb{F}_p)$-module of rk 1 corresponds to $(\underline{\omega}^{\otimes k}, \varphi)$?

(Suffices to do for $k=1$, because the correspondence respects $\otimes$)

Notice that we have a naturally occurring $\pi_1(S_m^{\varsigma})$ representation on a $\mathbb{Z}/p^m\mathbb{Z}$ module: the étale quotient of kernel of p^m on the universal curve E

Consider $\pi : \quad E \xrightarrow{\deg p} E/H = E^{(p)}$.

Denote by $\pi^m : E \longrightarrow E^{(p)} \longrightarrow E^{(p^2)} \longrightarrow \cdots \longrightarrow E^{(p^m)}$
& by $\check{\pi}^m$ the dual isogeny : $E^{(p^m)} \longrightarrow E$

As π_m is degree p^m, $\check{\pi}^m \circ \pi^m = [p^m]$ & $\ker \check{\pi}^m \cdot \underbrace{\mathrm{Im}\, E}_{= \ker [p^m]}[p^m]$

Claim : $\ker \check{\pi}^m$ is the étale quotient of $\mathrm{Im}\, E[p^m]$.

Pf : It is flat & f.p. over S^s_m because $\check{\pi}^m$ is

Unramified because every field valued point of S^s_m is in char p & mod p, $E^{(p^i)}$ are all ordinary elliptic curves $\cong E^{(p^i)} = E \times_{k,\sigma} k \times_{k,\sigma} k \times \cdots \times_{k,\sigma} k$

π^m is just F^m & $\check{\pi}^m$ is V^m
Frobenius ... verscheibung

Ordinary elliptic curves are characterized by $\ker V^m$ being étale, $\therefore$ so is $\ker \check{\pi}^m$

Moreover, F^m is purely inseparable, so if we don't quotient by $\ker \pi_m$, can't possibly get anything étale.

<u>Lemma</u> : The representation of $\pi_1(S^s_m)$ on $\ker \check{\pi}^m$ extends to a representation of $\pi_1(\bar{S}^s_m)$, i.e., it is "unramified at ∞".

Pf : STS for $m=1$ (Topological invariance of étale site)
Let K be function field of S^s_1
Want to show that inertia group of $\mathrm{Gal}(K^{sep}/K)$ at each cusp acts trivially on $\ker(V^m)$

$$\begin{tikzcd}
\mathrm{Gal}(K^{sep}/K) \arrow[r, two heads, "\theta"] \arrow[rrd] & \pi_1(S^s_1, \eta) \arrow[r] \arrow[rd] & \mathrm{Aut}(\ker V^m) \\
 & & \pi_1(\bar{S}^s_1, \eta) \arrow[u, dotted]
\end{tikzcd}$$

generic point

kernel is $\ker \theta$ + inertia group at cusps

As $\ker(V^m)$ is smooth affine $/\mathbb{F}_q$, suffices to check the action is trivial on each of its generic pts $\in (\ker V^m)_K$

As $(\ker V^m)_K$ is étale over K, $\therefore$ K^{sep} points are dense, & we can check action on $\ker V^m$ of $E^{(p^m)}_K(K^{sep})$

At the cusp $k((q))$ $(k = \mathbb{F}_q)$, inverse image of E is $T(q^n)/k((q))$
$E^{(p^m)} = T(q^{np^m})$ & $\check{\pi}^m$ is the map $T(q^{np^m}) \longrightarrow T(q^n)$ given by division by $\langle q^n \rangle$.

As all points of $\langle q^n \rangle$ are rational, the entire decomposition group acts trivially.

$\boxed{\text{THEOREM}}$: The rep of $\pi_1(\bar{S}^s_m)$ on $\ker \check{\pi}^m$ corresponds to $(\underline{\omega}, \varphi)$

Idea of proof :

- STS for S^s_m as restriction of $\pi_1(\bar{S}^s_m)$ reps to $\pi_1(S^s_m)$ is fully faithful

- Take a finite étale cover T of S_m trivializing the representation, so that $(\mathbb{Z}/p^m\mathbb{Z})_T \xrightarrow{\sim} (\ker \check{\pi}^m)_T$. Each point of $(\ker \check{\pi}^m)_T$ gives a map $(\mathbb{Z}/p^m\mathbb{Z})_T \longrightarrow (\ker \check{\pi}^m)_T$

- By Cartier duality $(\ker \pi^m)_T \longrightarrow (\mu_{p^m})_T \hookrightarrow (\mathbb{G}_m)_T$

an invariant differential coming from E $\xleftarrow{\text{pulls back}}$ $\frac{dt}{t}$

$$\Downarrow$$

We get a map $(\ker \check{\pi}^m)_T \longrightarrow \underline{\omega}_T$ inducing an isomorphism :

$$(\ker \check{\pi}^m)_T \otimes_{\mathbb{Z}/p^m\mathbb{Z}} \mathcal{O}_T \xrightarrow{\sim} \underline{\omega}_T$$

$\boxed{\text{THEOREM}}$: 1) $\pi_1(\bar{S}^s_m) \longrightarrow \text{Aut}(\ker(\check{\pi}^m)_T) \simeq (\mathbb{Z}/p^m\mathbb{Z})^\times$ is surjective

2) Restriction to $\pi_1(\mathcal{U})$ for $\mathcal{U}$ nonempty open $\subset \bar{S}^s_m$ remains surjective

Idea of proof : Let K = function field of S^s_1.

As before, STS $\text{Gal}(K^{sep}/K) \longrightarrow \text{Aut}(\ker V^m$ in $E^{(p^m)}(K^{sep}))$ is surjective

In fact, inertial group of G_K at any supersingular elliptic curve is already surjective.

Proof follows from the following theorem :

Let E, ω be an elliptic curve over $k[[A]]$ with Hasse invariant A, k being alg. closed of char p. Then the extension of $k((A))$ obtained by adjoining points of $\ker V^m$ is fully ramified of degree $p^{m-1}(p-1)$ with Galois group $(\mathbb{Z}/p^m\mathbb{Z})^\times$

This is proven by computing valuations of points in $\ker(V^m)$ in the formal group of E using Newton polygons.

§4: APPLICATIONS TO CONGRUENCES B/W MODULAR FORMS

COROLLARY: Let $k \in \mathbb{Z}$, $m \geq 1$, $p > 2$

TFAE:

1) $k \equiv 0 \mod (p-1)p^{m-1}$
2) k^{th} tensor power of the $\pi_1(\bar{S}_m^s)$-rep on the étale quotient of $\ker[p^m]$ is trivial
3) The sheaf $\underline{\omega}^{\otimes k}$ on $\bar{S}_m^s$ admits a nowhere vanishing section fixed by φ.
4) Over a non empty open $U \subset \bar{S}_m^s$, $\underline{\omega}^{\otimes k}$ admits a nowhere vanishing section fixed by φ
5) Over $\bar{S}_m^s$, $\underline{\omega}^{\otimes k}$ admits a section whose q-expansion at one of the cusps of $\bar{S}_m^s$ is identically 1.
6) Over a non empty open $U \subset \bar{S}_m^s$ which contains a cusp, $\underline{\omega}^{\otimes k}$ admits a section whose q-expansion at that cusp is identically 1.

Proof:

(1) $\Leftrightarrow$ (2):

$$\operatorname{Im} \pi_1(\bar{S}^s_m) = \operatorname{Aut}(\mathbb{Z}/p^m\mathbb{Z}) \simeq (\mathbb{Z}/p^m\mathbb{Z})^\times$$

$\therefore$ k tensor power is trivial

$\Leftrightarrow$ k is an exponent of $(\mathbb{Z}/p^m\mathbb{Z})^\times$

$\Leftrightarrow$ $k \equiv 0 \mod p^{m-1}(p-1)$

(2) $\Rightarrow$ (3):

$$\omega^{\otimes k}_{T'} = (\mathbb{Z}/p^m\mathbb{Z})\, \overset{\text{free generator}}{v} \otimes_{\mathbb{Z}/p^m\mathbb{Z}} \mathcal{O}_{T'}$$

$$\omega^{\otimes k} = (\mathbb{Z}/p^m\mathbb{Z}\, v \otimes \mathcal{O}_{T'})^{\pi_1(\bar{S}^s_m)}$$

$\pi_1(\bar{S}^s_m)$ acts trivially on v $\Rightarrow$ $\omega^{\otimes k} = \underbrace{\mathbb{Z}/p^m\mathbb{Z}\, v}_{\text{free of rk } 1} \otimes \mathcal{O}_{\bar{S}^s_m}$

$\Rightarrow$ $v \otimes 1$ is a section as required

(3) $\Rightarrow$ (2):

$$\underline{\omega}^{\otimes k} \cong \mathcal{O}_{\bar{S}^s_m}\, v \quad \leftarrow \text{the given section fixed by } \varphi$$

The representation of $\pi_1(\bar{S}^s_m)$ is obtained by taking an étale cover over which a φ-fixed basis exists & taking the global sections fixed by φ.

Here, the trivial étale cover works, so we get the trivial rep.

(3) $\Leftrightarrow$ (4) : restriction functor is fully faithful

(3) $\Rightarrow$ (5) : Using the explicit formula for φ

(5) $\Rightarrow$ (3) : Let f have q-expansion 1. $\varphi(f) - f$ has q expansion 0. By q-expansion principal for modular forms with growth condition $r=1$, we get that $\varphi(f) = f$

(4) $\Leftrightarrow$ (6) is same as (3) $\Leftrightarrow$ (5)

Let $U \subset \bar{S}^s_m$ be nonempty open containing a cusp.

If (1) holds, then we know there exists a nonvanishing φ-invariant section f on $\bar{S}^s_m$. Therefore, any φ-invariant non vanishing section of $\omega^{\otimes k}$ on U, g, must differ from $f|_U$ by a φ-invariant unit in $\mathcal{O}_U$, i.e. an element of $W_m(\mathbb{F}_p)^\times$. Therefore, g is extendable. The extension is unique because U contains a cusp + q-expansion principle.

$E_{p-1}^{k/(p-1)}$ is a nonvanishing section with q-expansion 1. By the above argument, all φ-invariant non-vanishing sections on U are $W_m(\mathbb{F}_p)^\times$ multiples of $E_{p-1}^{k/p-1}$.

COROLLARY: Suppose $f_i \in S(W(\mathbb{F}_q), 1, n, k_i)$ $i = 1,2$ & $k_1 \geq k_2$

Suppose q-expansions of f_1 & f_2 on at least one cusp of $\bar{M}_n(W(\mathbb{F}_q), 1)$ are congruent mod p^m & $f_1(q) \not\equiv 0$ mod p at that cusp.

Then $k_1 \equiv k_2$ mod $p^{m-1}(p-1)$

Pf : Reduce mod p^m. By hypothesis, f_1, f_2 are invertible in a nbhd U of the cusp, in some $\bar{S}^s_m$.

f_2/f_1 is an invertible section of $\underline{\omega}^{k_2-k_1}$ on U with q-expansion 1

By $(6 \Rightarrow 1)$, $k_2 - k_1 \equiv 0$ mod $p^{m-1}(p-1)$

COROLLARY: Let f be a true modular form of level n & wt k on $\Gamma_0(p)$, holomorphic at unramified cusps, and defined over fraction field K of $W(\mathbb{F}_q)$.

Suppose that each unramified cusp, all except the constant term of the q-expansion are in $W(\mathbb{F}_q)$. Then the constant terms of the q-expansions lie in $p^{-m}W(\mathbb{F}_q)$ where m is the largest integer s.t. $k \equiv 0$ mod $(p-1)p^{m-1}$

Pf : Let m_0 be min s.t. $p^{m_0}f$ has constant terms of q-expansions on unramified cusps in $W(\mathbb{F}_q)$

Let $g \in S(W(\mathbb{F}_q), 1, n, k)$ be defined as $g(E, \omega, \alpha_n, Y = E_p^{-1}) = p^{m_0} f(E, \omega, \alpha_n, H)$

Since m_0 is minimal, g has a q-expansion with constant term a unit u in $W(\mathbb{F}_q)$.

$u^{-1}g$ has q expansion $1 + p^{m_0}\sum_{i\geq 1} a_i q^i$. Mod p^{m_0}, $u^{-1}g$ has q-expansion 1.

By $(6) \Rightarrow (1)$, $k \equiv 0$ mod $p^{m_0-1}(p-1)$

§5. MODULAR FORMS OF WEIGHT χ

exponent of a generator

Let $\chi \in \mathrm{End}(\mathbb{Z}_p^\times) \cong \varprojlim \mathrm{End}(\mathbb{Z}/p^m\mathbb{Z})^\times \cong \varprojlim \mathbb{Z}/\varphi(p^m)\mathbb{Z}$

Consider $\rho : \pi_1(\bar{S}_m^S) \longrightarrow \mathrm{Aut}\left((\ker \check{\pi}_m)_T\right) \cong (\mathbb{Z}/p^m\mathbb{Z})^\times$ corresponding to $(\underline{\omega}, \varphi)$

ρ are compatible as m varies, & $\therefore$ so are $\chi \circ \rho$

Denote by $(\underline{\omega}^\chi, \varphi)$, the invertible sheaf corresponding to $\chi \circ \rho$. These are compatible as m varies.

Definition : A p-adic modular form of weight χ and level n, holomorphic at ∞, is a compatible family of global sections of $\underline{\omega}^\chi$ as m varies.

Remark 1: If $\chi = k \in \mathbb{Z} \subset \mathrm{End}(\mathbb{Z}_p^\times)$, we recover $S(W(\mathbb{F}_q), 1, n, k)$.
$\hookrightarrow \Leftrightarrow \chi \circ \rho \cong \rho^{\otimes k}$

Remark 2: For any χ, $(\underline{\omega}^\chi, \varphi)$ on $\bar{S}_m^S$ is isomorphic to $(\underline{\omega}^{\otimes k_m}, \varphi)$ for any $k_m \in \mathbb{Z}$ s.t. $k_m \equiv \chi \mod \varphi(p^m)$

Note: Suppose $k_m \equiv k_m' \equiv \chi$, then the isom between $(\omega^{\otimes k_m}, \varphi)$ & $(\omega^{\otimes k_m'}, \varphi)$ is given by multiplication by $E_{p-1}^{(k_m' - k_m)/p-1}$. This leaves q-expansions invariant mod p^m, & so we get a well defined & unique q-expansion of a p-adic modular form of wt χ.

THEOREM :

1) Let $\chi \in \mathrm{End}(\mathbb{Z}_p^\times)$, & f be a modular form of weight χ & level n, holomorphic at ∞, defined over $W(\mathbb{F}_q)$. Then $\exists$ a sequence of integers $0 \le k_1 \le k_2 \le k_3 \le \cdots$ s.t.

$$k_m \equiv \chi \mod \varphi(p^m)$$

and a sequence of true modular forms f_i of weight k_i & level n, holomorphic at ∞ s.t.

$$f_m \equiv f \mod p^m \quad \text{in } q\text{-expansion}$$

2) Conversely, let $\{k_m\}_{m \ge 1}$ be an <u>arbitrary</u> sequence of integers, and suppose given a sequence $f_m \in S(W(\mathbb{F}_q), 1, n, k_m)$ of p-adic modular forms of weights k_i s.t.

$$f_{m+1} \equiv f_m \mod p^m \quad \text{in } q\text{-expansion at each cusp}$$
$$\exists m \text{ s.t. } f_m \not\equiv 0 \mod p^m \text{ in } q\text{-expansion}$$

Then the sequence of weights k_m converges to $\chi \in \mathrm{End}(\mathbb{Z}_p^\times)$ & $\exists!$ modular form $f = \lim f_m$ of wt χ & level n, hol at ∞, s.t.

$$f_m \equiv f \mod p^m \quad \text{in } q \text{ expansion}$$

Outline of pf:

1) From definition.

2) • Multiply f_m by high powers of $E_{p-1}^{p^{m-1}}$ to get the weights in increasing order & positive

• Consider the limit q-expansion. $\exists$ max m_0 s.t. p^{m_0} divides the q-expansion.

Then $f_m = p^{m_0} g_m$ for $m > m_0$ where g_m is a wt k_m p-adic modular form w/ q-expansions $\not\equiv 0$ mod p, of

• Using the sequence $\{g_{m+m_0}\}$, we get a congruence relation on weights & they converge to χ.

• $p^{m_0} g_{m+m_0}$ mod p^m define a compatible family of sections of $\underline{\omega}^{\chi}$ on $\bar{S}_m^{s}$

COROLLARY:

Let $\chi \in \operatorname{End} \mathbb{Z}_p^{\times}$. Let $0 \le k_1 \le k_2 \le \cdots$ be a sequence of integers s.t. $k_m \equiv \chi \mod \varphi(p^m)$

Let f_m be a sequence of true modular forms of weight k_m & level n on $\Gamma_0(p)$, hol at the unramified cusps & defined over $\operatorname{Frac} W(\mathbb{F}_q) = K$.

Suppose the non constant terms of all the q expansions of f_m are in $W(\mathbb{F}_q)$ & at each cusp

$$f_{m+1}(q) - f_{m+1}(0) \equiv f_m(q) - f_m(0) \mod p^m$$

Then:

1) if $\chi \neq 0$, let m_0 be the largest integer s.t. $\chi \equiv 0 \mod \varphi(p^{m_0})$. Then for $m \geq m_0$, $p^{m_0} f_m$ has integral q-expansions.

2) Further, at each cusp, we have the congruence on constant terms: $p^{m_0} f_{m+1}(0) \equiv p^{m_0} f_m(0) \mod p^{m-m_0}$ for all $m > m_0$.

Pf:

1) is as before

2) Let $h_m := p^{m_0} f_{m+1} - p^{m_0} f_m \underbrace{E_{p-1}^{(k_{m+1}-k_m)/p-1}}_{\uparrow}$

padding up f_m to get the same weight as f_{m+1}

mod p^m, $h_m(q) \equiv h_m(0)$

$\frac{h_m}{p^m}$ is a modular form of weight k_m with nonconstant q-expansion coefficients in $W(\mathbb{F}_q)$. As before p^{m_0} gives a bound on the denominator $\Rightarrow$ $p^{m-m_0} \mid h_m$

$\Rightarrow$ $p^{m_0} f_{m+1}(0) \equiv p^{m_0} f(0) \mod p^{m-m_0}$

EXAMPLE :

Take $f_m = G_{k_m}$ with q-expansion given by $\displaystyle -\frac{b_{k_m}}{2k_m} + \sum_{n\geq 1} \sigma_{k_m-1}(n)\, q^n$

Choose k_m to be strictly increasing with m (forces $k_m - 1 \geq m$)
s.t. they converge to a desired χ

Claim : $f_{m+1}(q) - f_{m+1}(0) \equiv f_m(q) - f_m(0) \mod p^m$

Pf :

coef of q^n in LHS − RHS is :

$$\underbrace{\sum_{\substack{d\mid n\\(d,p)=1}} \left(d^{k_{m+1}-1} - d^{k_m-1}\right)}_{\substack{= d^{k_m-1}\left(d^{k_{m+1}-k_m} - 1\right)\\ \equiv 0 \mod p^m}} + \underbrace{\sum_{\substack{d\mid n\\(d,p)\neq 1}} \left(p^{k_{m+1}-1}\left(\frac{d}{p}\right)^{k_{m+1}-1} - p^{k_m-1}\left(\frac{d}{p}\right)^{k_m-1}\right)}_{\substack{= p^{k_m-1}(\qquad)\\ \equiv 0 \mod p^m}}$$

↑
d is a unit mod p^m &
$k_{m+1} \equiv k_m \mod \varphi(p^m)$

∴ By the result earlier, $\lim p^{m_0} f_m \overset{\text{def}}{=} p^{m_0} G^*_\chi$ is a modular form of wt χ. The nonconstant part of the q-expansion is given by $\sum_{n\geq 1} a_n q^n$, where

$$a_n = p^{m_0} \lim_m \Big(\sum_{\substack{d\mid n\\(d,p)=1}} d^{k_m-1} + \underbrace{\sum_{\substack{d\mid n\\(d,p)\neq 1}} p^{k_m-1}\cdot\left(\frac{d}{p}\right)^{k_m-1}}_{\equiv 0} \mod p^m \Big)$$

$$= p^{m_0} \lim_m \sum_{\substack{d\mid n\\(d,p)=1}} \frac{d^{k_m}}{d} = p^{m_0} \sum_{\substack{d\mid n\\(d,p)=1}} \frac{\chi(d)}{d}$$

∴ "G^*_χ" has q-expansion with coef of q^n = $\displaystyle\sum_{\substack{d\mid n\\(d,p)=1}} \frac{\chi(d)}{d}$

Note that even if χ is an even positive integer $2k \geq 4$, $G^*_\chi \neq G_{2k}$. In particular, the coef of q^n for the latter is

$$\sum_{d\mid n} d^{2k-1} = \sum_{d\mid n} \frac{\chi(d)}{d} \neq \sum_{\substack{d\mid n\\(d,p)=1}} \frac{\chi(d)}{d}$$

↑
when $p \mid n$