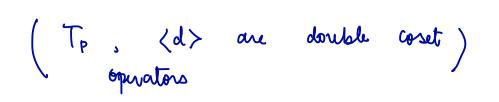
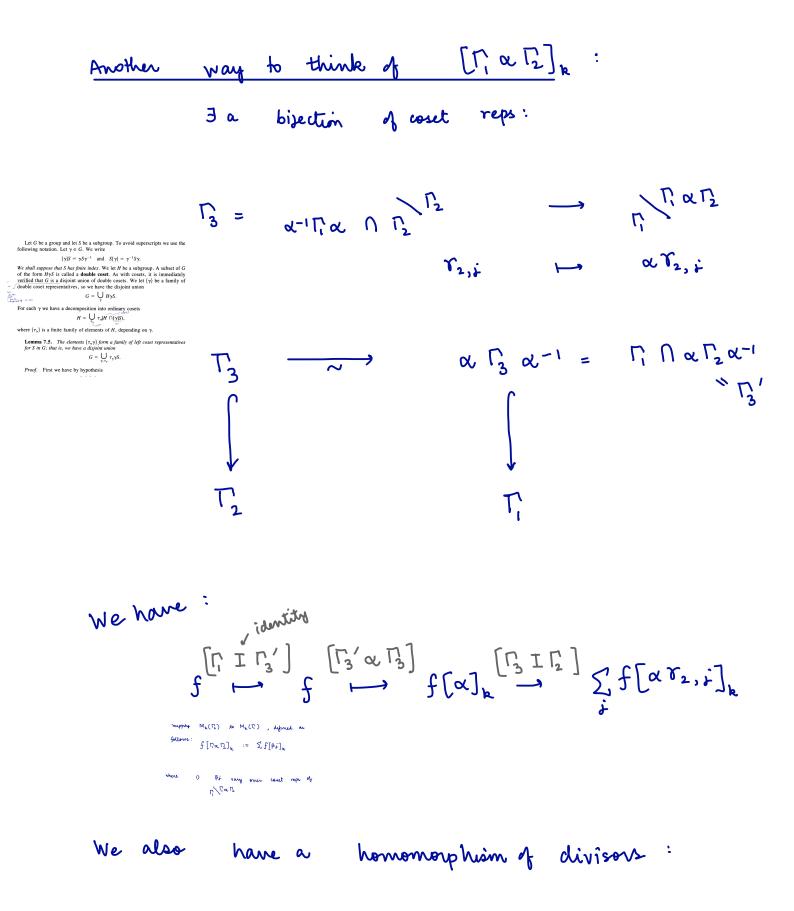
Part 1: Hecke operators

OUTLINE:  
1) Hecke operators are double coset op.  
2) Double coset op induce multivalued for called  
correspondences  
3) Correspondences induce maps on cohomologies  
Double coset operatore: Let 
$$\alpha \in GL_2(\mathbb{Q})^+$$
  
 $\Pi, \Pi_2 \subset SL_2(\mathbb{Z})$  be Congruence subgps.  
 $[\Gamma \alpha \Gamma_2]_k$  is an operator  
mapping  $M_k(\Gamma_2)$  to  $M_k(\Gamma)$ , dynaed as  
follows:  
 $f[\Pi \alpha \Gamma_2]_k := \sum f[\beta_i]_k$ 

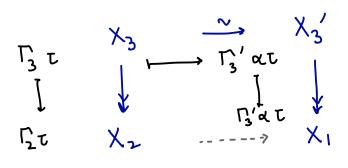
where i) 
$$\beta j$$
 vary over coset reps of  
 $\Gamma_{1} \setminus \Gamma_{1} \propto \Gamma_{2}$   
2)  $f[\beta j]_{k} (T) := (dut \beta j)^{k-1} (\frac{1}{cT+d})^{k} f(\beta jT)$   
 $\int [\alpha \ b]_{cd}$ 

Example : Let 
$$\alpha = \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix}$$
  $\Gamma_{1} = \Gamma_{2} = \Gamma_{0}(N)$   
P(N. Then  $\{\begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix}$ ,  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\}$   
give caset reps:  $f_{1} = \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix}$ ,  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\}$   
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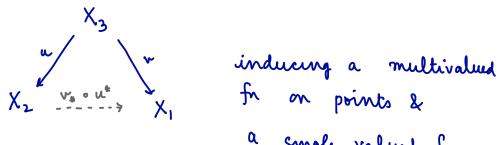




Let Xibe Ti\H\*



so, Hecke operator may be vierned as a "correspondence"



inducing a multivalued fr on points & a single valued fr on divisors by summing over. We have an induced map:

$$H^{\circ}(X_{1}, \Omega') \xrightarrow{u_{\bullet} v^{*}} H^{\circ}(X_{2}, \Omega')$$

Aside: pushfind along u works approximately the following way:  $\tilde{u} = UUi$   $\omega$  I  $I^{h}$   $\tilde{f}_{i}$  I  $u = \omega_{al inverse}$   $\Sigma_{i}^{i} (h_{i}^{-1})^{*} \omega|_{Ui}$ Worry about ramification pts & glueing

Note that:  $S_2(\Gamma_i) \cong \Omega'_{hol}(X_i)$ Fact: The above map agrees with the map induced on  $S_2(\Gamma_i)$  by the action of  $[\Gamma_i \propto \Gamma_2]$  on modular forms of wt 2

Somehow this map on cohomology is supposed to generalize the same way although g don't know how prohforwards work generally.

Part 2	:	Cohomo	Lognz	of	locally	symn	retric	spaces
In what	fo	llows,	all	man	nifolds	are	C∞	
L=R	or C							
E f.d.	V-5.	/L						
C°°(M	; E	) :	Spo		l em	oth fr		/ values
<b>A<sup></sup>°</b> (M	; E	) :			I sm encor pr			
				∕°V ·	T*(M)	8		E) J product Joundle
			~= ~~ ~~	.N°V	T*(M)	) <sub>Ør</sub>	E	

Theorem: 
$$H^*(\Gamma, E)$$
 is canonically isom to  $H^*(A(X, E)^{\Gamma})$ 

Sketch q pf  

$$\Gamma \setminus X$$
 is a smooth manifold, X is contractible  
 $\Rightarrow \Gamma \setminus X$  is Eilenberg Maclane Space  $K(\Gamma, L)$   
 $approxitly$   
The is supposed to imply  
 $H^*(\Gamma; E) = H^*(\Gamma \setminus X; \tilde{E})$   
 $\downarrow$  vector bundle  
 $X \times E/(r,c) \sim (T,x, p(T),c)$   
 $\downarrow$   
 $\downarrow$  bet  $\pi : X \rightarrow \Gamma \setminus X$   
 $A(\Gamma \setminus X; \tilde{E}) \rightarrow A(X; E)^{\Gamma}$   
 $\downarrow = g(T)(\omega(T^{T}, T^{T}))$   
 $\downarrow = g(T) = g(T) = g(T)$   
 $\downarrow = g(T) = g(T) = g(T)$ 

Cohomology 
$$q A(T \setminus x_s \tilde{E}) \cong$$
  
 $H^*(T \setminus x_s \tilde{E})$  by some  
application  $q$  de Rham's Thm.

Let  $\mathbf{I}^{\infty}(\mathbf{E}) = \mathbf{C}^{\infty}(\mathbf{G}, \mathbf{E})^{\Gamma}$  $\int f \in \mathbf{C}^{\infty}(\mathbf{G}, \mathbf{E}): f(\mathbf{T} \cdot \mathbf{g}) = g(\mathbf{T}) \cdot f(\mathbf{g})$ 

(This is precisely the induced rep in the C<sup>$$\infty$$</sup> sense  
& G-action is given by rt-translation  
 $(9 \cdot f)(3) = f(39)$ 

 $A^{q}(G; E) = Hom(\Lambda^{q} \mathcal{G}, C^{\infty}(\mathcal{G}, E))$ 

Let 
$$\omega \in A^{q_{\ell}}(G; E)^{r_{\ell}}$$
  
 $\Leftrightarrow \qquad \omega(Y \cdot g, Y) \in A^{q_{\ell}} \partial f$   
position  
 $e_{Y} \cdot at$   
 $= g(Y) \cdot \omega(g, Y)$ 

$$\omega(-,Y) \in \mathbb{I}^{\infty}(E)$$

$$\therefore \ \omega \in \operatorname{Hom}(\Lambda^{\operatorname{v}}\mathcal{O}_{F}, \mathbb{I}^{\infty}(E))$$

$$\operatorname{Clearly}, \ \operatorname{He} \ \operatorname{converse} \ \operatorname{holds} \ \operatorname{as} \ \operatorname{mell}.$$

$$\operatorname{and}_{L: \ A^{\operatorname{v}}(G;E)}^{\Gamma} \cong \operatorname{Hom}(\Lambda^{\operatorname{v}}\mathcal{O}_{F}; \mathbb{I}^{\circ}(E))$$

PROPOSITION :

(Here, Kactson Nor/k via adjè rep & on I<sup>∞</sup>(E) via rt translation)

WOTE is F-invariant as wis. c(wote) is K-invariant under rt translation & is annihilated by interior products ix for x e R as x proches forward to the O vector field. Err...

1° co-compact => G is nec. unimodular

=> T\G has a unique rt G invariant Radon measure, nonzero on non-empty open sets.

$$\therefore \text{ this is a fr on } \bigcap G & \text{ we can} \\ define a global & calar product by \\ \text{integrating over } \bigcap G & \text{ let } I_2(E) & \text{ be} \\ \text{integrating over } \bigcap G & \text{ let } I_2(E) & \text{ be} \\ & \text{integrating over } \bigcap G & \text{ let } I_2(E) & \text{ inder this} \\ & \text{ the completion of } I^{\infty}(E) & \text{ inder this} \\ & \text{ is cont.} & \text{ scalar product } \\ & \text{ is cont.} & \text{ is cont.} & \text{ on } \bigcap G & \text{ can be realized} \\ & \text{ where } f \in I^{\infty}(E) & \text{ is } \\ & \text{ where } f \in I^{\infty}(E) & \text{ is } \\ & \text{ where } f \in I^{\infty}(E) & \text{ is } \\ & \text{ where } f \in I^{\infty}(E) & \text{ is } \\ \end{array}$$

je .......

I<sup>2</sup>(E) is a unitary Cr-mod w.r.t. right translations

Apparently 
$$(I^{2}(E))^{\infty}_{j} = I^{\infty}(E)$$
  
via a smooth vec  $\ll g \mapsto \pi(g)v$  is smooth

By some thm of Gelfand & Piatetski-Shapiro-

$$I^{2}(E) = \bigoplus_{\substack{\pi \text{ irred} \\ G_{1} - rep}} m(\pi, T', E) H_{\pi}$$

$$\Rightarrow \qquad \mathbb{I}^{\infty}(\mathbb{E}) = (\mathbb{I}^{2}(\mathbb{E}))^{\infty} = \left( \bigoplus_{\pi \in \mathcal{G} \text{ incp}}^{\wedge} \mathbb{I}_{\pi}, \Gamma, \mathbb{E} \right) \mathbb{H}_{\pi} \right)^{\infty}$$

$$\begin{array}{rcl} \hline \text{Theorem} : & H^{*}(\Gamma, E) \stackrel{\sim}{=} \bigoplus & m(\pi, \Gamma, E) H^{*}(\mathfrak{P}, K; H^{\infty}_{\pi}) \\ & \pi \in Girrep & & & \\ \hline \text{This sum is finite.} & & & & \\ \hline \text{The natural how } j^{*} : H^{*}(\mathfrak{P}, K; E^{\Gamma}) \longrightarrow & & \\ \hline \text{The natural how } j^{*} : H^{*}(\mathfrak{P}, K; E^{\Gamma}) \longrightarrow & & \\ \hline \text{H}^{*}(\mathfrak{P}, K; T^{\infty}(E^{\Gamma})) & & \\ \hline \text{Induced by } E^{\Gamma} \xrightarrow{\sim} T^{\infty}(E^{\Gamma}) & \\ \hline \text{He space of } & \\ \hline \text{e} \mapsto (\mathfrak{P} \mapsto e) & \\ \hline \text{K-finite } & \\ \hline \text{vectors } & \\ \hline \text{os image is the contribution of the trivial } & \\ \hline \text{H}_{10} & \\ \hline \text{He}_{10} & \\ \hline \text{He}_{10$$

Sketch of proof: 
$$p_{H^*}(\sigma_t, K; I^{\infty}(E))$$
  
 $H^*(\Gamma; E) = H^*(\sigma_t, K; (\widehat{\Phi} m(\pi, \Gamma, E) H_{\pi})^{\infty})$ 

Let 
$$C_{S}^{*} = \bigoplus_{\pi \in S} \operatorname{Hom}_{K}(\Lambda^{*} \sigma_{F}/R, m(\pi, \Gamma, E) \operatorname{H}_{\pi}^{*})$$

$$C_{S'}^* = Hom_k (\Lambda^* \sigma_f / k ; (\hat{\Theta}_{\pi \in S^c} - M_{\pi} H_{\pi})^{\infty})$$

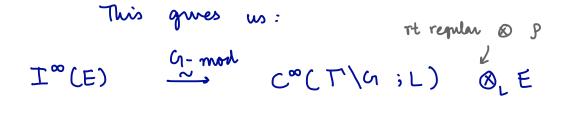
Then 
$$H^{*}(\Gamma; E) = \bigoplus_{\pi \in S} m_{\pi} H^{*}(\sigma_{F}, K'; H_{\pi}^{\infty})$$
  
 $\bigoplus H^{*}(C_{S'}^{*})$ 

$$H^{*}(\Gamma; E) = H^{*}(\Gamma \setminus X, \tilde{E})$$
  
Support & locally contractible

This is supposed to imply dim H°(T; E) < 00

∴ I exists a finite set SC Ginep s.t. H\* for TF & S is O. We now want to show that H\* (C\*s,) = 0

Assume (p, E) is the restriction to T qa rep of G. Consider the map  $C^{\infty}(G, E) \rightarrow C^{\infty}(G, E)$  $f \mapsto (q \vdash g(q^{-1}) f(q))$ f f is T invariant, then F is defined m $<math>\Gamma \setminus G$ 



f ⊢ F (g ↦ g(g) F(g)) ∽ F

<sup>9</sup>f(p, E) comes from  $\alpha$  rep of G, we have  $H^{ov}(A^*(G; E)^{\Gamma}) \cong H^*(O; C^{ov}(\Gamma \setminus (G, L) \otimes E)$ 

Similar to before, we have  

$$L^{2}(\Gamma \setminus G) = \bigoplus_{\substack{\pi \in G \text{ integ}}}^{\infty} m(\pi, \Gamma) H_{\pi}$$

$$C^{\infty}(\Gamma \setminus G) = (L^{2}(T \setminus G))^{\infty}$$

$$T^{\infty}(E) \cong (\bigoplus_{\substack{\pi \in G \text{ integ}}}^{\infty} m_{\pi} H_{\pi})^{\infty} \otimes E$$

Theorem:  

$$H^*(\Gamma; E) = \bigoplus m_{\pi} H^*(\mathcal{T}, K; H^{\infty}_{\pi} \otimes E)$$
  
The natural homomorphism  $J^*: H^*(\mathcal{T}, K; E)$   
 $\longrightarrow H^*(\mathcal{T}, K; C^{\infty}(\Gamma \setminus G; L) \otimes E) = H^*(\Gamma; E)$   
(induced by identification of  $E$  w/ the space of  
constant  $E$ -valued from  $\Gamma \setminus G$   
is injective. Its image to the contribution of  
the trivial rep of  $G$ .