Part 1: Hecke operators

OUTLINE:

1) Hecke operators are double coset op.
2) Double coset op induce multivalued fns called correspondences
3) Correspondences induce maps on cohomologies

Double coset operators: Let $\alpha \in GL_2(\mathbb{Q})^+$

$\Gamma_1, \Gamma_2 < SL_2(\mathbb{Z})$ be congruence subgps.

$[\Gamma_1 \alpha \Gamma_2]_k$ is an operator mapping $M_k(\Gamma_2)$ to $M_k(\Gamma_1)$, defined as follows:

$$f[\Gamma_1 \alpha \Gamma_2]_k := \sum_j f[\beta_j]_k$$

where 1) β_j vary over coset reps of $\Gamma_1 \backslash \Gamma_1 \alpha \Gamma_2$

2) $$f[\beta_j]_k(\tau) := (\det \beta_j)^{k-1} \left(\frac{1}{c\tau + d}\right)^k f(\beta_j \tau)$$

$$\beta_j = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

<u>Example</u> : let $\alpha = \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix}$ $\Gamma_1 = \Gamma_2 = \Gamma_0(N) \to \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$ mod N

$p \nmid N$. Then $\left\{ \underbrace{\begin{bmatrix} 1 & j \\ 0 & p \end{bmatrix}}_{\beta_j}, \underbrace{\begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}}_{\beta_\infty} \right\}$ $j \in [0, p-1)$

give coset reps of $\Gamma_0(N) \backslash \Gamma_0(N) \alpha \Gamma_0(N)$

$$\mathbb{Z}\langle \beta_j \tau, 1 \rangle = \frac{1}{p} \underbrace{\mathbb{Z}\langle \tau + j, p \rangle}_{\Lambda_j}$$

$$\mathbb{Z}\langle \beta_\infty \tau, 1 \rangle = \mathbb{Z}\langle p\tau, 1 \rangle = \Lambda_\infty$$

Λ_j, Λ_∞ → sublattice of index p

- $f[\beta_j]_k(\tau) = p^{k-1}\left(p^{-k} f(\beta_j \tau)\right)$

Λ_j corresponds to $(E_{\beta_j \tau}, p\omega)$

$$\therefore f(\Lambda_j) = f(E_{\beta_j \tau}, p\omega)$$
$$= p^{-k} f(\beta_j \tau, \omega)$$

$$\therefore f[\beta_j]_k(\tau) = p^{k-1} f(\Lambda')$$

$$\therefore [\Gamma_0(N) \alpha \Gamma_0(N)]_k = T_p \Big|_{M_k(\Gamma_0(N))}$$

(T_p, $\langle d \rangle$ are double coset operators)

Another way to think of $[\Gamma_1 \alpha \Gamma_2]_k$:

$\exists$ a bijection of coset reps :

$$\Gamma_3 = \alpha^{-1}\Gamma_1\alpha \cap \Gamma_2 \backslash \Gamma_2 \longrightarrow \Gamma_1 \backslash \Gamma_1 \alpha \Gamma_2$$

$$\gamma_{2,j} \longmapsto \alpha\gamma_{2,j}$$

Let G be a group and let S be a subgroup. To avoid superscripts we use the following notation. Let $\gamma \in G$. We write

$$[\gamma]S = \gamma S \gamma^{-1} \quad \text{and} \quad S[\gamma] = \gamma^{-1} S \gamma.$$

We shall suppose that S has finite index. We let H be a subgroup. A subset of G of the form $H\gamma S$ is called a **double coset**. As with cosets, it is immediately verified that G is a disjoint union of double cosets. We let $\{\gamma\}$ be a family of double coset representatives, so we have the disjoint union

$$G = \bigcup_{\gamma} H\gamma S.$$

For each γ we have a decomposition into ordinary cosets

$$H = \bigcup_{\tau_\gamma} \tau_\gamma (H \cap [\gamma]S),$$

where $\{\tau_\gamma\}$ is a finite family of elements of H, depending on γ.

Lemma 7.5. *The elements $\{\tau_\gamma\gamma\}$ form a family of left coset representatives for S in G; that is, we have a disjoint union*

$$G = \bigcup_{\gamma, \tau_\gamma} \tau_\gamma \gamma S.$$

Proof. First we have by hypothesis

. . . .

$$\begin{tikzcd}
\Gamma_3 \arrow[r, "\sim"'] \arrow[d, hook] & \alpha \Gamma_3 \alpha^{-1} = \Gamma_1 \cap \alpha \Gamma_2 \alpha^{-1} =: \Gamma_3' \arrow[d, hook] \\
\Gamma_2 & \Gamma_1
\end{tikzcd}$$

We have :

$$f \overset{[\Gamma_1 \, I \, \Gamma_3']}{\longmapsto} f \overset{[\Gamma_3' \alpha \Gamma_3]}{\longmapsto} f[\alpha]_k \overset{[\Gamma_3 \, I \, \Gamma_2]}{\longrightarrow} \sum_j f[\alpha\gamma_{2,j}]_k$$

identity

mapping $M_k(\Gamma_2)$ to $M_k(\Gamma_1)$, defined as follows :

$$f[\Gamma_1 \alpha \Gamma_2]_k := \sum f[\beta_j]_k$$

where 1) β_j vary over coset reps of $\Gamma_1 \backslash \Gamma_1 \alpha \Gamma_2$

We also have a homomorphism of divisors :

let X_i be $\Gamma_i \backslash \mathcal{H}^*$

$$\begin{tikzcd}
\Gamma_3\tau \arrow[d, mapsto] & X_3 \arrow[d] \arrow[r, mapsto] \arrow[rr, bend left, "\sim"] & \Gamma_3'\alpha\tau \arrow[d, mapsto] & X_3' \arrow[d] \\
\Gamma_2\tau & X_2 \arrow[rr, dashed] & \Gamma_3'\alpha\tau & X_1
\end{tikzcd}$$

$$\begin{tikzcd}
\sum_j [\Gamma_3 \gamma_{2,j}\tau] \arrow[r] & \sum_j [\Gamma_3'\alpha\gamma_{2,j}\tau] \arrow[d, "\text{pushforward}"] \\
{[\Gamma_3\tau]} \arrow[u, "\text{pullback}"] & \sum [\Gamma_3'\alpha\gamma_{2,j}\tau]
\end{tikzcd}$$

degree of divisor · by deg of the map

So, Hecke operator may be viewed as a "correspondence"

$$\begin{tikzcd}
& X_3 \arrow[dl, "u"'] \arrow[dr, "v"] & \\
X_2 \arrow[rr, dashed, "v_* \circ u^*"] & & X_1
\end{tikzcd}$$

inducing a multivalued fn on points & a single valued fn on divisors by summing over.

We have an induced map:

$$H^0(X_1, \Omega^1) \xrightarrow{u_* v^*} H^0(X_2, \Omega^1)$$

[Aside: pushfwd along u works approximately the following way:

$$\begin{array}{ccc} \tilde{U} = \cup U_i & & \omega \\ \downarrow h \quad \nearrow h_i^{-1}, \text{ a local inverse} & & \mapsto \\ U & & \sum_i (h_i^{-1})^* \omega|_{U_i} \end{array}$$

]

Worry about ramification pts & glueing

Note that : $S_2(\Gamma_i) \cong \Omega^1_{hol}(X_i)$

Fact : The above map agrees with the map induced on $S_2(\Gamma_i)$ by the action of $[\Gamma_1 \alpha \Gamma_2]$ on modular forms of wt 2

Somehow this map on cohomology is supposed to generalize the same way although I don't know how pushforwards work generally.

Part 2 : Cohomology of locally symmetric spaces

In what follows, all manifolds are C^∞

$L = \mathbb{R}$ or $\mathbb{C}$

E f.d. v.s. / L

$C^\infty(M ; E)$: Space of smooth fns w/ values in E

$A^q(M ; E)$: Space of smooth sections of the tensor product bundle

$$\Lambda^q T^*(M) \otimes \underbrace{(M \times E)}_{\text{trivial product bundle}}$$

$$\underset{\text{as f.d.}}{\cong} \Omega^q T^*(M) \otimes_{\mathbb{R}} E$$

Let G be a Lie gp w/ finitely many connected components,

K, a maximal compact subgroup.

$X = G/K$

Γ = a discrete, torsion free subgp of G ($\therefore$ free action on G/K)

(ρ, E) a f.d. real or complex rep of Γ

Fact: X is homeomorphic to Euclidean space

Further $\Gamma \backslash X$ is a locally symmetric space

(i.e. Riemannian manifold s.t. each x has a nbhd U & a diffeo s_x of U that is an isometry inverting tangent vectors at x)

outline:

1) relationship b/w cohomology of Γ, cohomology of $\Gamma \backslash X$ & relative Lie algebra cohomology

2) Decomposition of cohomology when Γ is co-compact &
- E is unitary Γ-mod
- E is a G-mod

Theorem: $H^*(\Gamma, E)$ is canonically isom to $H^*(A(X, E)^\Gamma)$

Sketch of pf

$\Gamma \backslash X$ is a smooth manifold, X is contractible

$\underset{\text{apparently}}{\Rightarrow}$ $\Gamma \backslash X$ is Eilenberg Maclane Space $K(\Gamma, 1)$

This is supposed to imply

$$H^*(\Gamma; E) = H^*(\Gamma \backslash X; \tilde{E})$$

vector bundle

$$X \times E / (x,e) \sim (\gamma \cdot x, \rho(\gamma) \cdot e)$$

Let $\pi : X \longrightarrow \Gamma \backslash X$

$$A(\Gamma \backslash X; \tilde{E}) \longrightarrow A(X; E)^\Gamma$$

$$\omega \longmapsto \omega \circ \pi$$

$(\gamma \cdot \omega)(x, y_1, \dots, y_q) := \rho(\gamma)(\omega(\gamma^{-1}m, \gamma^{-1}y_1, \dots \gamma^{-1}y_q))$

pt

vector fields

$$\Gamma(\tilde{E}) \otimes \Omega^q(\Gamma \backslash X)$$

$f_i \otimes \omega_i$

smooth fns $X \to E$ s.t. $f_i(\gamma \cdot x) = \rho(\gamma) \cdot f_i(x)$

Cohomology of $A(\Gamma \backslash X; \tilde{E}) \cong H^*(\Gamma \backslash X; \tilde{E})$ by some application of de Rham's Thm.

Let $I^\infty(E) = C^\infty(G, E)^\Gamma$

$\{ f \in C^\infty(G, E) : f(\gamma \cdot g) = \rho(\gamma) \cdot f(g) \}$

(This is precisely the induced rep in the C^∞ sense

& G-action is given by rt- translation

$(g \cdot f)(\xi) = f(\xi g)$)

$$A^q(G;E) = \mathrm{Hom}(\Lambda^q \mathfrak{g}, C^\infty(G,E))$$

Let $\omega \in A^q(G;E)^\Gamma$

$\Leftrightarrow \quad \omega(\underbrace{\gamma \cdot g}_{\substack{\text{position} \\ \text{ev. at}}}, Y)$ $\quad Y \in \Lambda^q \mathfrak{g}$

$= \rho(\gamma) \cdot \omega(g, Y)$

$\omega(-, Y) \in I^\infty(E)$

$\therefore \omega \in \mathrm{Hom}(\Lambda^q \mathfrak{g}, I^\infty(E))$

Clearly, the converse holds as well.

and

$\iota : A^q(G;E)^\Gamma \cong \mathrm{Hom}(\Lambda^q \mathfrak{g}; I^\infty(E))$

PROPOSITION:

Let $\pi : G \to G/K$ be the canonical projection.

$\exists$ an isom of graded complexes:

$$A^*(X;E)^\Gamma \longrightarrow \mathrm{Hom}_K(\Lambda^* \mathfrak{g}/\mathfrak{k} \,;\, I^\infty(E))$$

$$\omega \longmapsto \underset{\substack{\cap \\ A^*(G;E)^\Gamma}}{\omega \circ \pi} \longmapsto \iota(\omega \circ \pi)$$

cohomology

$$\therefore \quad H^*(\Gamma; E) \cong H^*(\mathfrak{g}, K \,;\, I^\infty(E))$$

(Here, K acts on $\Lambda^* \mathfrak{g}/\mathfrak{k}$ via adj't rep & on $I^\infty(E)$ via rt translation)

[$\omega \circ \pi$ is Γ-invariant as ω is.
$\iota(\omega \circ \pi)$ is K-invariant under rt translation & is annihilated by interior products i_x for $x \in \mathfrak{k}$ as x pushes forward to the 0 vector field. Err...]

Γ CO-COMPACT

E UNITARY Γ- module

$L = \mathbb{C}$

Γ co-compact $\Rightarrow$ G is nec. unimodular

$\Rightarrow$ $\Gamma \backslash G$ has a unique rt G invariant Radon measure, nonzero on non-empty open sets.

If $u, v \in I^\infty(E)$

Then $(u(\gamma \cdot x), v(\gamma \cdot x))_E \overset{I^\infty(E) \text{ definition}}{=} (\gamma \cdot u(x), \gamma \cdot v(x))_E$

$$\underset{\text{unitary}}{=} (u(x), v(x))_E$$

$\therefore$ this is a fn on $\Gamma \backslash G$ & we can define a global scalar product by integrating over $\Gamma \backslash G$. Let $I_2(E)$ be the completion of $I^\infty(E)$ under this scalar product.

measurable $\because x \mapsto u(x), v(x)$ $\downarrow$ $(\ \ )_E$ is cont.

certainly finite norm $\because$ measure on $\Gamma \backslash G$ can be realized by integrating over a cpct fund. domain, where $f \in I^\infty(E)$ is bdd.

$I^2(E)$ is a unitary G-mod w.r.t. right translations

Apparently $\underbrace{(I^2(E))^\infty}_{v \text{ is a smooth vec} \iff g \mapsto \pi(g)v \text{ is smooth}} = I^\infty(E)$

By some thm of Gelfand & Piatetski-Shapiro

$$I^2(E) = \hat{\bigoplus_{\substack{\pi \text{ irred} \\ G\text{-rep}}}} \underbrace{m(\pi, \Gamma, E)}_{\in \mathbb{N}} H_\pi$$

$$\Rightarrow \quad I^\infty(E) = (I^2(E))^\infty = \Big(\hat{\bigoplus_{\pi \in G \text{ irrep}}} m(\pi, \Gamma, E) H_\pi \Big)^\infty$$

$\boxed{\text{Theorem}}$: • $H^*(\Gamma, E) \cong \bigoplus_{\pi \in G \text{ irrep}} m(\pi, \Gamma, E) H^*(\mathfrak{g}, K; H_\pi^\infty)$

I think may be substituted by $H_{\pi,0}$, the space of K-finite vectors as image $\Lambda\mathfrak{g}/R$ is perhaps nec. contained in $H_{\pi,0}$. What of smoothness?

This sum is finite.

• The natural hom $j^* : H^*(\mathfrak{g}, K; E^\Gamma) \longrightarrow H^*(\mathfrak{g}, K; I^\infty(E))$ $= H^*(\Gamma, E)$ induced by $E^\Gamma \xrightarrow{\sim} I^\infty(E)^G$, $e \mapsto (g \mapsto e)$

is injective.

Its image is the contribution of the trivial rep π_0 of G, and $m(\pi_0, \Gamma, E) = \dim E^\Gamma$

Sketch of proof :

$$H^*(\Gamma; E) \overset{\text{prop 2 pages ago}}{=} H^*(\mathfrak{g}, K; I^\infty(E)) = H^*(\mathfrak{g}, K; (\hat{\bigoplus} m(\pi, \Gamma, E) H_\pi)^\infty)$$

Let $S \subset G$ irrep be finite.

Let $$C_S^* = \bigoplus_{\pi \in S} \mathrm{Hom}_K(\Lambda^* \mathfrak{g}/\mathfrak{k}, m(\pi, \Gamma, E) H_\pi^\infty)$$

$$C_{S'}^* = \mathrm{Hom}_K(\Lambda^* \mathfrak{g}/\mathfrak{k} ; (\hat{\bigoplus_{\pi \in S^c}} m_\pi H_\pi)^\infty)$$

Then $$H^*(\Gamma; E) = \bigoplus_{\pi \in S} m_\pi H^*(\mathfrak{g}, K; H_\pi^\infty) \oplus H^*(C_{S'}^*)$$

$$H^*(\Gamma; E) = H^*(\underbrace{\Gamma \backslash X}_{\text{cpct \& locally contractible}}, \tilde{E})$$

This is supposed to imply $\dim H^*(\Gamma; E) < \infty$

$\therefore$ $\exists$ exists a finite set $S \subset G$ irrep s.t. H^* for $\pi \notin S$ is 0. We now want to show that

$H^*(C_{S'}^*) = 0$

Γ - CO COMPACT

E - G-MODULE

Assume (ρ, E) is the restriction to Γ of a rep of G.

Consider the map $C^\infty(G, E) \longrightarrow C^\infty(G, E)$

$$f \longmapsto \left(g \overset{F}{\longmapsto} \rho(g^{-1}) f(g)\right)$$

If f is Γ invariant, then F is defined on $\Gamma\backslash G$

This gives us:

rt regular $\otimes$ ρ

$$I^\infty(E) \xrightarrow[\sim]{G\text{-mod}} C^\infty(\Gamma\backslash G\,; L) \otimes_L E$$

$$f \longmapsto F$$

$$(g \mapsto \rho(g) F(g)) \longleftarrow F$$

If (ρ, E) comes from a rep of G, we have

$$H^q(A^*(G; E)^\Gamma) \cong H^*(\mathfrak{g}; C^\infty(\Gamma\backslash G, L) \otimes E)$$

Similar to before, we have

$$L^2(\Gamma\backslash G) = \widehat{\bigoplus_{\pi \in G \text{ irrep}}} m(\pi,\Gamma) H_\pi$$

$$C^\infty(\Gamma\backslash G) = (L^2(\Gamma\backslash G))^\infty$$

$$I^\infty(E) \cong \Big(\widehat{\bigoplus_{\pi\in G \text{ irrep}}} m_\pi H_\pi \Big)^\infty \otimes E$$

Theorem:

$$H^*(\Gamma; E) = \bigoplus m_\pi H^*(\mathfrak{g}, K; H_\pi^\infty \otimes E)$$

The natural homomorphism $j^* : H^*(\mathfrak{g}, K; E)$

$$\longrightarrow H^*(\mathfrak{g}, K; C^\infty(\Gamma\backslash G; L) \otimes E) = H^*(\Gamma; E)$$

(induced by identification of E w/ the space of constant E-valued fns on $\Gamma\backslash G$)

is injective. Its image is the contribution of the trivial rep of G.