SMOOTHNESS OF COMPONENTS OF THE EMERTON-GEE STACK FOR GL_2

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ABSTRACT. Let K be a finite unramified extension of $\mathbb{Q}_p$, where $p > 2$. [CEGS19] and [EG22] construct a moduli stack of two dimensional mod p representations of the absolute Galois group of K. We show that most irreducible components of this stack (including several non-generic components) are isomorphic to quotients of smooth affine schemes. We also use this quotient presentation to compute global sections on these components.

CONTENTS

1. Introduction

Let $K/\mathbb{Q}_p$ be a finite extension. Following [CEGS19, § 3], define the stack $\mathcal{R}$ to be the moduli stack of étale φ-modules of rank two. By [Fon91], there is an equivalence of categories between étale φ-modules and p-adic Galois representations of $G_\infty := \mathrm{Gal}(\overline{K}/K_\infty)$ allowing us to view $\mathcal{R}$ as a moduli stack of said representations.

The theory of Breuil-Kisin modules developed in [Kis06] gives us a way to associate to any Breuil-Kisin module an étale φ-module. Indeed, by denoting the moduli stack of rank two Breuil-Kisin modules of finite height h by $\mathcal{C}_h$, then there is a morphism $\mathcal{C}_h \to \mathcal{R}$ given (topologically) by

$$\mathfrak{M} \longmapsto \mathfrak{M}[1/u],$$

where u is a formal variable. The stack of Breuil-Kisin modules of height 1, denoted $\mathcal{C}_1$, admits a scheme-theoretic image $\mathcal{Z}_1$ via the above morphism. Breuil-Kisin modules of height 1 correspond to étale φ-modules which admit natural extensions to representations of $G_K = \mathrm{Gal}(\overline{K}/K)$ so the substack $\mathcal{Z}_1$ may be viewed as a moduli stack of representations of G_K.

Such stacks are of increasing interest in the study of p-adic Galois representations, see for example [EG22]. One method to study such objects is to introduce Breuil-Kisin modules with descent datum. This approach allows the consideration of those Galois representations that arise from generic fibers of finite flat group schemes after restriction to a finite tamely ramified extension of K. In fact, all representations except très ramifiée ones arise this way. Here, the très ramifiée representations are the twists of certain extensions of the trivial character by the mod p cyclotomic character.

Let K' be a tamely ramified extension of K. Endowing our Breuil-Kisin modules and étale φ-modules with descent datum from K' to K allows us to define the morphism $\mathcal{C}_1^{dd} \to \mathcal{R}^{dd}$. To focus on those non très ramifiée representations, we enforce a Barsotti-Tate condition on points in the scheme-theoretic image. The Barsotti-Tate condition on representations with coefficients in a characteristic p field turns out to correspond to a strong determinant condition on Breuil-Kisin modules, see [CEGS19, Theorem 3.9.2]. Let $\mathcal{C}_1^{dd,\mathrm{BT}}$ denote the stack of Breuil-Kisin modules of height 1 with descent data that satisfy the strong determinant condition. With this, we attain a morphism $\mathcal{C}_1^{dd,\mathrm{BT}} \to \mathcal{R}^{dd}$ whose scheme-theoretic image is denoted $\mathcal{Z}_1^{dd}$, the stack of non très ramifiée G_K-representations. We will take these stacks to be defined over $\mathbb{F}$, a finite field extension of $\mathbb{F}_p$. To reduce notation, we will suppress the decorations dd and 1 in the symbols for our stacks with the assumption that all of our objects have descent data and correspond to height 1 Breuil-Kisin modules. Both $\mathcal{C}^{\mathrm{BT}}$ and $\mathcal{Z}$ are algebraic stacks of finite presentation.

1.1. **Main result.** By [CEGS19, § 4], the irreducible components of $\mathcal{Z}$ can be described in terms of non-Steinberg Serre weights. Indeed, for such a Serre weight σ, the associated irreducible component $\mathcal{Z}(\sigma)$ is such that the $\overline{\mathbb{F}}_p$-points of $\mathcal{Z}(\sigma)$ are those representations having σ as a Serre weight. Such irreducible components serve as the main objects of study in this paper. In particular, our main result is a smooth presentation of the irreducible component $\mathcal{Z}(\sigma)$ and a computation of its global functions.

Theorem (Theorem 5.0.1)**.** *Let $p > 2$. Let K be an unramified extension of $\mathbb{Q}_p$ of degree f with residue field k. Let $\mathcal{Z}(\sigma)$ be the irreducible component of $\mathcal{Z}$ indexed by a non-Steinberg Serre weight $\sigma = \sigma_{\vec{a},\vec{b}} = \bigotimes_{i=0}^{f-1}(\det^{a_i} \operatorname{Sym}^{b_i} k^2) \otimes_{k,\kappa_i} \mathbb{F}$, where $\{\kappa_i\}_{i=0}^{f-1}$ is the set of the distinct embeddings of k into $\mathbb{F}$, and $\kappa_{i+1}^p = \kappa_i$. Suppose σ satisfies the following properties:*

(1) $\vec{b} \neq (0, 0, \ldots, 0)$.
(2) $\vec{b} \neq (p-2, p-2, \ldots, p-2)$.
(3) *Extend the indices of b_i's to all of $\mathbb{Z}$ by setting $b_{i+f} = b_i$. Then $(b_i)_{i\in\mathbb{Z}}$ does not contain a contiguous subsequence of the form $(0, p-2, \ldots, p-2, p-1)$ of length ≥ 2.*

Then $\mathcal{Z}(\sigma)$ is smooth and isomorphic to a quotient of $\mathrm{GL}_2 \times \mathrm{SL}_2^{f-1}$ by $\mathbb{G}_m^{f+1} \times \mathbb{G}_a^f$. The ring of global functions of $\mathcal{Z}(\sigma)$ is isomorphic to $\mathbb{F}[x, y][\frac{1}{y}]$.

Note that the case $\vec{b} = (0, 0, \ldots, 0)$ can be studied using Fontaine-Lafaille methods. The key utility of this paper is in providing a description of components indexed by non-Fontaine-Lafaille Serre weights.

1.2. **Outline of the article.** We begin in Section 2, by providing a concrete classification of certain Breuil-Kisin modules defined over Artinian local $\mathbb{F}$-algebras with descent data classified by a tame inertial $\mathbb{F}$-type τ. This closely follows the ideas of [CDM18] which describe a method to classify such Breuil-Kisin modules into one of three forms which are convenient for computations. We then describe the automorphisms of such Breuil-Kisin modules that are needed to study the stack $\mathcal{C}^{\mathrm{BT}}$. We find that this method is not foolproof however, and identify *obstruction* conditions on τ for which our methods fail.

In Section 3, for a given tame intertial $\mathbb{F}$-type τ, we identify an irreducible component $\mathcal{X}(\tau)$ of $\mathcal{C}^{\mathrm{BT}}$ that can be written as a quotient of smooth affine schemes. This utilizes the classification from Section 2. We then use this quotient presentation to compute the global functions on $\mathcal{X}(\tau)$.

Finally, Section 4 is where we analyze the relationship between $\mathcal{X}(\tau)$'s and the irreducible components of $\mathcal{Z}$. We first explicitly identify the irreducible component of $\mathcal{Z}$ that turns out to be the image of $\mathcal{X}(\tau) \subset \mathcal{C}^{\mathrm{BT}}$. Subject to the aforementioned obstruction conditions, we allow τ to vary in order to obtain as many irreducible components of $\mathcal{Z}$ as possible in the images of $\mathcal{X}(\tau)$'s. We then show that the map from $\mathcal{X}(\tau)$ to $\mathcal{Z}$ is fully faithful, and conclude finally that $\mathcal{X}(\tau)$ is isomorphic to its scheme-theoretic image.

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1.4. **Notation and conventions.** Fix a prime $p > 2$. Let K be a finite, unramified extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_K$ and residue field k. Let $f := f(K/\mathbb{Q}_p)$. Upon fixing an algebraic closure $\overline{K}$, we let $G_K = \mathrm{Gal}(\overline{K}/K)$ denote the absolute Galois group of K and define I_K and I_K^{w} to be the inertia and wild inertia subgroups of G_K respectively. Fix a uniformizer π of K and a p-power compatible sequence $(\pi_n)_{n\geq 0}$ so that $\pi_0 = \pi$ and $\pi_{n+1}^p = \pi_n$. We define the field K_∞ to be the compositum $K_\infty = \bigcup_n K(\pi_n)$ with associated Galois group $G_\infty = \mathrm{Gal}(\overline{K}/K_\infty) \subset G_K$.

Serving to capture our descent data, let K'/K be a totally tamely ramified extension of degree $e := p^f - 1$ obtained by adjoining an e-th root of p to K which we denote $\pi_{K'}$. Let $k' = k$ be the residue field of K'. Denote the minimal polynomial of $\pi_{K'}$ over K by $E(u) = u^e - p$. $\mathrm{Gal}(K'/K)$ is cyclic of order e, and is isomorphic to $\mu_e(K)$, the group of eth roots of unity in $W(k') = \mathcal{O}_K$. This isomorphism is given by $h : \mathrm{Gal}(K'/K) \to \mu_e(K)$, defined by $h(g) = \frac{g(\pi_{K'})}{\pi_{K'}}$.

Serving as coefficients, let $\mathbb{F}$ be a finite extension of $\mathbb{F}_p$ which we assume to be large enough such that $\mathbb{F}$ contains all embeddings of k into $\overline{\mathbb{F}}_p$. We fix an embedding $\kappa_0 : k \hookrightarrow \mathbb{F}$, and recursively define $\kappa_i : k \hookrightarrow \mathbb{F}$ for any $i \in \mathbb{Z}$ such that $\kappa_{i+1}^p = \kappa_i$ and so $\kappa_{i+f} = \kappa_i$ for any $i \in \mathbb{Z}$. Since there are f such embeddings, we will commonly take the index to be $i \in \mathbb{Z}/f\mathbb{Z}$.

1.4.1. *Serre Weights.* Recall $k/\mathbb{F}_p$ is a degree f extension. A *Serre weight* is an isomorphism class of irreducible $\mathbb{F}$-representations of $\mathrm{GL}_2(k)$. Any such representation is, up to isomorphism, of the form

$$\sigma_{\vec{a},\vec{b}} := \bigotimes_{i=0}^{f-1} (\det{}^{a_i} \operatorname{Sym}^{b_i} k^2) \otimes_{k,\kappa_i} \mathbb{F},$$

where $0 \leq a_i, b_i \leq p-1$ and not all a_i equal to $p-1$. We say $\sigma_{\vec{a},\vec{b}}$ is *Steinberg* if each b_i equals $p-1$.

1.4.2. *Tame Inertial $\mathbb{F}$-Types.* An *inertial $\mathbb{F}$-type* is (the isomorphism class of) a representation $\tau : I_K \to \mathrm{GL}_2(\mathbb{F})$ with open kernel which extends to G_K. An inertial $\mathbb{F}$-type is called *tame* if $\tau|_{I_K^{\mathrm{w}}}$ is trivial.

Let $\tau : I_K \to \mathrm{GL}_2(\mathbb{F})$ be a tame inertial $\mathbb{F}$-type. Then $\tau \cong \eta \oplus \eta'$, where $\eta, \eta' : I_K \to \mathbb{F}^\times$ are tamely ramified characters. We say τ is a *tame principal series $\mathbb{F}$-type* if η, η' both extend to characters of G_K. We will assume that the tame inertial $\mathbb{F}$-types τ factor via $I(K'/K) = \mathrm{Gal}(K'/K)$.

Given $\tau \cong \eta \oplus \eta'$, let γ_i be the unique integer in $[0, p^f - 1)$ such that $\eta\eta'^{-1}(g) = \kappa_i \circ h(g)^{\gamma_i}$. We also define $z_j \in \{0, \dots, p-1\}$ for $j \in \mathbb{Z}/f\mathbb{Z}$ so as to satisfy

$$\gamma_i = \sum_{j=0}^{f-1} z_{i-j} p^j. \tag{1.4.1}$$

Note that this implies $p\gamma_i = z_{i+1}e + \gamma_{i+1}$. At times, we will assume the indexing set for z_i to be $\mathbb{Z}$ via the natural quotient map $\mathbb{Z} \to \mathbb{Z}/f\mathbb{Z}$.

2. Breuil-Kisin modules with descent

In this section, we introduce some basic definitions and properties about Breuil-Kisin modules and their moduli space. To begin, we introduce the relevant notions which we will use for the rest of the paper. Once again, while the general definition of Breuil-Kisin modules works over characteristic 0, we will restrict to characteristic p.

Let $W(k')$ denote the ring of Witt vectors of k' and define $\mathfrak{S} := W(k')[\![u]\!]$. The ring $\mathfrak{S}$ is equipped with a Frobenius endomorphism φ such that $u \mapsto u^p$ which is semilinear with respect to the usual (arithmetic) Frobenius on $W(k')$ lifted from the natural Frobenius on k'. The Galois group $\mathrm{Gal}(K'/K)$ acts on $\mathfrak{S}$ via $g(\sum a_i u^i) = \sum a_i h(g)^i u^i$, where $g \in \mathrm{Gal}(K'/K)$, $\sum a_i u^i \in \mathfrak{S}$ and $h : \mathrm{Gal}(K'/K) \to W(k')$ is the map given by $h(g) = \frac{g(\pi_{K'})}{\pi_{K'}}$. For later convenience, we also let $v = u^e$.

Let R be an $\mathbb{F}$-algebra. We let $\mathfrak{S}_R := (W(k') \otimes_{\mathbb{Z}_p} R)[\![u]\!]$ be the extension of scalars equipped with R-linear actions of φ and $\mathrm{Gal}(K'/K)$ naturally extended from the φ and $\mathrm{Gal}(K'/K)$ actions on $\mathfrak{S}$. Let $\mathfrak{e}_i$ denote the idempotent of $W(k') \otimes_{\mathbb{Z}_p} R \cong k' \otimes_{\mathbb{F}_p} R \cong \prod_{\kappa_i : k' \hookrightarrow \mathbb{F}} R$ corresponding to $(0, \dots, 0, 1, 0, \dots, 0)$ with 1 in the ith coordinate. We can then write

$$\mathfrak{S}_R \cong \bigoplus_{i \in \mathbb{Z}/f\mathbb{Z}} R[\![u]\!]\mathfrak{e}_i.$$

Definition 2.0.1. *A Breuil-Kisin module with R-coefficients and descent data from K' to K is a triple $(\mathfrak{M}, \varphi_{\mathfrak{M}}, \{\widehat{g}\}_{g \in \mathrm{Gal}(K'/K)})$, consisting of a finitely generated projective $\mathfrak{S}_R$-module $\mathfrak{M}$ such that*

- *$\mathfrak{M}$ admits a φ-semilinear map $\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$ such that the induced map $\Phi_{\mathfrak{M}} = 1 \otimes \varphi_{\mathfrak{M}} : \mathfrak{S}_R \otimes_{\varphi, \mathfrak{S}_R} \mathfrak{M} \to \mathfrak{M}$ is an isomorphism after inverting $E(u)$.*
- *$\mathfrak{M}$ admits an additive bijection $\widehat{g} : \mathfrak{M} \to \mathfrak{M}$ for each $g \in \mathrm{Gal}(K'/K)$ which commutes with $\varphi_{\mathfrak{M}}$, respects the group structure $\widehat{g_1 \circ g_2} = \widehat{g_1} \circ \widehat{g_2}$, and satisfies $\widehat{g}(sm) = g(s)\widehat{g}(m)$ for all $s \in \mathfrak{S}_R$ and $m \in \mathfrak{M}$.*

We say $\mathfrak{M}$ has *height* at most $h \geq 0$ if the cokernel of $\Phi_{\mathfrak{M}}$ is killed by $E(u)^h$. We say a Breuil-Kisin module $\mathfrak{M}$ is rank d if the underlying $\mathfrak{S}_R$-module has constant rank d. A morphism of Breuil-Kisin modules with R-coefficients and descent data is a morphism of $\mathfrak{S}_R$-modules that commutes with the φ-action and the Galois action.

Localizing a Breuil-Kisin module $\mathfrak{M}$ by inverting u gives rise to an étale-φ module which we define below.

Definition 2.0.2. *An étale-φ module with R-coefficients and descent data from K' to K is a triple $(M, \varphi_M, \{\widehat{g}\}_{g\in \mathrm{Gal}(K'/K)})$, consisting of a finitely generated projective $\mathfrak{S}_R[1/u]$-module M such that*

- *M admits a φ-semilinear map $\varphi_M : M \to M$ such that the induced map $\Phi_M = 1 \otimes \varphi_M : \mathfrak{S}_R[1/u] \otimes_{\varphi, \mathfrak{S}_R[1/u]} M \to M$ is an isomorphism.*
- *M admits an additive bijection $\widehat{g} : M \to M$ for each $g \in \mathrm{Gal}(K'/K)$ which commutes with φ_M, respects the group structure $\widehat{g_1 \circ g_2} = \widehat{g_1} \circ \widehat{g_2}$, and satisfies $\widehat{g}(sm) = g(s)\widehat{g}(m)$ for all $s \in \mathfrak{S}_R[1/u]$ and $m \in M$.*

Let $\mathfrak{M}$ be a Breuil-Kisin module with R-coefficients. We can decompose $\mathfrak{M}$ in terms of idempotents

$$\mathfrak{M} = \bigoplus_{i=0}^{f-1} \mathfrak{M}_i,$$

where $\mathfrak{M}_i = \mathfrak{e}_i\mathfrak{M}$. Similarly, let M be an étale-φ module with R-coefficients. We can decompose M in terms of idempotents

$$M = \bigoplus_{i=0}^{f-1} M_i,$$

where $M_i = \mathfrak{e}_i M$.

In this paper, we will be using this decomposition by idempotents repeatedly and without further comment.

2.1. **Inertial descent datum.** Let $\mathfrak{M}$ (resp. M) be a Breuil-Kisin module (resp. étale-φ module) of rank two with R coefficients. Let $\tau = \eta \oplus \eta'$ be a fixed tame principal series $\mathbb{F}$-type.

Definition 2.1.1. *We say $\mathfrak{M}$ has tame principal series $\mathbb{F}$-type τ if (Zariski locally on* $\operatorname{Spec} R$ *if necessary) there exists a* $\mathrm{Gal}(K'/K)$*-equivariant isomorphism $\mathfrak{M}_i/u\mathfrak{M}_i \cong R\eta \oplus R\eta'$ for each i.*

Definition 2.1.2. *An inertial basis of $\mathfrak{M}_i$ (resp. M_i) is an ordered basis with respect to which the Galois action is given diagonally by $\eta \oplus \eta'$.*

Base change matrices that switch a set of inertial bases (comprising a basis for each $\mathfrak{M}_i$, or M_i as the case may be) to another set of inertial bases will be called inertial base change matrices, and the corresponding change of bases will be called an inertial base change.

Our first order of business is to show that we can always find an inertial basis for a Breuil-Kisin module of tame $\mathbb{F}$-type τ.

Lemma 2.1.3. *Let $\mathfrak{M}$ be a Breuil-Kisin module of tame $\mathbb{F}$-type τ. For all $i \in \mathbb{Z}/f\mathbb{Z}$, (Zariski locally on R if necessary), there exists an ordered $R[\![u]\!]$-basis (e_i, f_i) of $\mathfrak{M}_i$ such that the action of*

$g \in \mathrm{Gal}(K'/K)$ is given by

$$\widehat{g}(e_i) = \eta(g)e_i, \quad \widehat{g}(f_i) = \eta'(g)f_i. \tag{2.1.1}$$

Proof. Let $\{\xi_j\}_j$ be the set of pairwise distinct characters of $\mathrm{Gal}(K'/K)$ valued in $\mathbb{F}$-algebras.

Fix $i \in \mathbb{Z}/f\mathbb{Z}$. We first claim that as $\mathbb{F}[\![v]\!][\mathrm{Gal}(K'/K)]$-modules, $\mathfrak{M}_i$ decomposes as $\bigoplus_{\xi_j} \mathfrak{M}_{i,\xi_j}$, where $\mathfrak{M}_{i,\xi_j}$ is the part of $\mathfrak{M}_i$ on which $\mathrm{Gal}(K'/K)$ acts via the character ξ_j. Granting this claim, we observe further that this decomposition is in fact as $R[\![v]\!][\mathrm{Gal}(K'/K)]$-modules. This is because each $\mathfrak{M}_{i,\xi_j}$ is an $R[\![v]\!][\mathrm{Gal}(K'/K)]$-submodule, since multiplying an eigenvector by something in $R[\![v]\!]$ doesn't change the eigencharacter.

We now consider this decomposition mod u. Without loss of generality (up to passage to an affine open cover of $\operatorname{Spec} R$ if necessary), we have the following isomorphism of $R[\mathrm{Gal}(K'/K)]$ modules:

$$Rx \oplus Ry \cong \bigoplus_{\xi_j} \mathfrak{M}_{i,\xi_j} \mod u,$$

where x is an eigenvector for $\mathrm{Gal}(K'/K)$ with eigencharacter η and y is an eigenvector for $\mathrm{Gal}(K'/K)$ with eigencharacter η'. Therefore, there exists some $e_i \in \mathfrak{M}_{i,\eta}$ whose reduction mod u is x, and similarly, there exists some $f_i \in \mathfrak{M}_{i,\eta'}$ whose reduction mod u is y.

Suppose $(\sum_{k\geq 0} a_k u^k)e_i + (\sum_{k\geq 0} b_k u^k)f_i = 0$ for some $\sum_{k\geq 0} a_k u^k, \sum_{k\geq 0} b_k u^k \in R[\![u]\!]$. Suppose n is the smallest degree so that either a_n or b_n is nonzero. As $\mathfrak{M}_i$ is u-torsion free, we can divide the equation by u^n and assume, without loss of generality, that $a_0 \neq 0$. As e_i and f_i are linearly independent mod u, a_0 is forced to be 0, giving a contradiction. Therefore, e_i and f_i are linearly independent, and we have an inclusion $R[\![u]\!]e_i \oplus R[\![u]\!]f_i \hookrightarrow \mathfrak{M}_i$, which is an equality mod u. By Nakayama, $R[\![u]\!]e_i \oplus R[\![u]\!]f_i \hookrightarrow \mathfrak{M}_i$ is an equality.

Now, we prove the claim that every $\mathbb{F}[\![v]\!][\mathrm{Gal}(K'/K)]$ module decomposes as a direct sum over characters.

Let $\mathfrak{x} \in \mathbb{F}[\![v]\!][\mathrm{Gal}(K'/K)] \subset \mathbb{F}(\!(v)\!)[\mathrm{Gal}(K'/K)]$. As the order of $\mathrm{Gal}(K'/K)$ is prime to the characteristic of $\mathbb{F}$ and $\mathrm{Gal}(K'/K)$ is abelian, $\mathfrak{x} = \mathfrak{x}_1 + ... + \mathfrak{x}_r$ where each $\mathfrak{x}_j \in \mathbb{F}(\!(v)\!)[\mathrm{Gal}(K'/K)]$ is acted upon via the character ξ_j. We induct over r to show that each $\mathfrak{x}_j$ is in fact an element of $\mathbb{F}[\![v]\!][\mathrm{Gal}(K'/K)]$. Suppose the statement is true for $r = n$. Given $\mathfrak{x} = \mathfrak{x}_1 + ... + \mathfrak{x}_n + \mathfrak{x}_{n+1}$, there exists $g \in \mathrm{Gal}(K'/K)$ such that $\xi_1(g) \neq \xi_2(g)$. Applying g to $\mathfrak{x} = \mathfrak{x}_1 + ... + \mathfrak{x}_n + \mathfrak{x}_{n+1}$, we get $g\mathfrak{x} = \xi_1(g)\mathfrak{x}_1 + ... + \xi_{n+1}(g)\mathfrak{x}_{n+1} \in \mathbb{F}[\![v]\!][\mathrm{Gal}(K'/K)]$. As each $\xi_j(g)$ is an eth root of unity, it is a unit in R, and therefore $\mathfrak{x}_1 + \xi_1^{-1}(g)\xi_2(g)\mathfrak{x}_2 + ... + \xi_1^{-1}(g)\xi_{n+1}(g)\mathfrak{x}_{n+1}$ is an element of $\mathbb{F}[\![v]\!][\mathrm{Gal}(K'/K)]$. Subtracting $\mathfrak{x}_1 + ... + \mathfrak{x}_n + \mathfrak{x}_{n+1}$, we obtain a nonzero sum with n terms and the induction hypothesis applies.

This argument shows that if A is a free $\mathbb{F}[\![v]\!][\mathrm{Gal}(K'/K)]$ module, it breaks up as direct sum over characters ξ_j, pairwise distinct. Let B be a submodule of A. Suppose $\mathfrak{x}_1 + ... + \mathfrak{x}_r \equiv 0 \mod B$ where each $\mathfrak{x}_j$ is an eigenvector with eigencharacter ξ_j. The exact same argument as in the previous paragraph shows that each $\mathfrak{x}_j \equiv 0 \mod B$. Therefore B also breaks up as direct sum over characters, and consequently, so does A/B. This establishes the claim. $\square$

The Frobenius $\Phi_{\mathfrak{M}}$ restricts to a map $\varphi^*\mathfrak{M}_{i-1} \to \mathfrak{M}_i$ which we will call the i-th Frobenius map and denote by $\Phi_{\mathfrak{M},i}$. After fixing an inertial basis for each i, let the i-th Frobenius map be represented by

$$F_i = \begin{pmatrix} A_1^{(i)} & A_2^{(i)} \\ A_3^{(i)} & A_4^{(i)} \end{pmatrix},$$

such that $A_j^{(i)} \in R[\![u]\!]$ with

$$\Phi_{\mathfrak{M},i}(1 \otimes e_{i-1}) = A_1^{(i)} e_i + A_3^{(i)} f_i, \quad \Phi_{\mathfrak{M},i}(1 \otimes f_{i-1}) = A_2^{(i)} e_i + A_4^{(i)} f_i,$$

for any $i \in \mathbb{Z}/f\mathbb{Z}$.

Lemma 2.1.4. *Suppose $\eta \neq \eta'$. After fixing an inertial basis for each i, each Frobenius linearization $\Phi_{\mathfrak{M},i} : \varphi^*(\mathfrak{M}_{i-1}) \to \mathfrak{M}_i$ has a matrix of the form:*

$$F_i = \begin{pmatrix} s_1^{(i)} & u^{e-\gamma_i} s_2^{(i)} \\ u^{\gamma_i} s_3^{(i)} & s_4^{(i)} \end{pmatrix}, \tag{2.1.2}$$

where $s_j^{(i)} \in R[\![v]\!]$, for $i \in \mathbb{Z}/f\mathbb{Z}$.

Proof. This follows easily from the commutative condition between φ and $\mathrm{Gal}(K'/K)$ actions. □

We will further refine the form of the Frobenius action in a subsequent section but first, we must find a description of inertial base change matrices.

Lemma 2.1.5. *Suppose $\mathfrak{M}$ (resp. M) is a Breuil-Kisin module (resp. étale-φ module) with an inertial basis for each $\mathfrak{M}_i$ (resp. M_i).*

For each i, let $P_i = \begin{pmatrix} b_1^{(i)} & b_2^{(i)} \\ b_3^{(i)} & b_4^{(i)} \end{pmatrix}$ be an inertial base change matrix in $\mathrm{GL}_2(R[\![u]\!])$ (resp. $\mathrm{GL}_2(R(\!(u)\!))$). Then $b_1^{(i)}, b_4^{(i)} \in R(\!(v)\!)$, $b_2^{(i)} \in u^{e-\gamma_i} R(\!(v)\!)$ and $b_3^{(i)} \in u^{\gamma_i} R(\!(v)\!)$.

Proof. P_i is an inertial base change matrix if and only if for all $g \in \mathrm{Gal}(K'/K)$, we have:

$$P_i^{-1} \cdot \begin{pmatrix} \eta(g) & 0 \\ 0 & \eta'(g) \end{pmatrix} \cdot g(P_i) = \begin{pmatrix} \eta(g) & 0 \\ 0 & \eta'(g) \end{pmatrix} \iff$$
$$\begin{pmatrix} \eta(g) g(b_1^{(i)}) & \eta(g) g(b_2^{(i)}) \\ \eta'(g) g(b_3^{(i)}) & \eta'(g) g(b_4^{(i)}) \end{pmatrix} = \begin{pmatrix} \eta(g) b_1^{(i)} & \eta'(g) b_2^{(i)} \\ \eta(g) b_3^{(i)} & \eta'(g) b_4^{(i)} \end{pmatrix}.$$

Comparing the entries and letting $\chi = \eta\eta'^{-1}$, we get $g(b_1^{(i)}) = b_1^{(i)}$, $g(b_2^{(i)}) = \chi(g)^{-1} b_2^{(i)}$, $g(b_3^{(i)}) = \chi(g) b_3^{(i)}$ and $g(b_4^{(i)}) = b_4^{(i)}$. The statement of the lemma follows immediately. □

2.2. **Stacks.** We introduced several stacks of Breuil-Kisin modules and Galois representations in Section 1. In this section, we recall more precisely the definitions of some of these stacks from [CEGS19] for later reference. We will be suppressing the superscripts "*dd*" (for descent data) and "1" (indicating that the stack is defined over $\mathbb{F}$) from the original notation used in [CEGS19].

Definition 2.2.1. *We define $\mathcal{C}$ to be the stack defined over $\mathbb{F}$ characterized by the following property: For an $\mathbb{F}$-algebra R, $\mathcal{C}(R)$ is the groupoid of rank two Breuil-Kisin modules defined over R of height at most one and descent data from K' to K.*

Definition 2.2.2. *We define $\mathcal{C}^{\tau}$ to be the closed substack of $\mathcal{C}$ corresponding to Breuil-Kisin modules of tame inertial $\mathbb{F}$-type τ.*

Definition 2.2.3. *We define $\mathcal{C}^{\mathrm{BT}}$ (resp. $\mathcal{C}^{\tau,\mathrm{BT}}$) to be the closed substack of $\mathcal{C}$ (resp. $\mathcal{C}^{\tau}$) corresponding to Breuil-Kisin modules that additionally satisfy the the strong determinant condition in* [CEGS19, Definition 3.5.2].

We will not recall the precise definition of the strong determinant condition because it is technical and not important for this article. The essential idea is that if $\mathbb{F}'$ is a finite extension of $\mathbb{F}$, then by [CEGS19, Lemma 3.5.16], the $\mathbb{F}'$ points of $\mathcal{C}^{\mathrm{BT}}$ are precisely those Breuil-Kisin modules whose corresponding Galois representations become Barsotti-Tate over K'. The $\mathcal{C}^{\mathrm{BT}}$ is the reduced closure of such points.

By inverting the formal power series variable u in $\mathfrak{S}$, we can transform a Breuil-Kisin module into an étale-φ module. This gives us a morphism from $\mathcal{C}$ to the stack of étale-φ modules with descent data, denoted $\mathcal{R}$.

Definition 2.2.4. *We define $\mathcal{Z}$ (resp. $\mathcal{Z}^{\tau}$) to be the scheme-theoretic image of the natural morphism $\mathcal{C}^{\mathrm{BT}} \to \mathcal{R}$ (resp. $\mathcal{C}^{\tau,\mathrm{BT}} \to \mathcal{R}$), in the sense of* [EG21].

By [CEGS19, Thm 3.9.2], the $\overline{\mathbb{F}}_p$-points of $\mathcal{Z}$ are the continuous representations $\bar{r} : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ that are not très ramifiée. The irreducible components of $\mathcal{Z}$ are labelled by Serre weights, so that if σ is a Serre weight and $\mathcal{Z}(\sigma)$ is the corresponding irreducible component, then the $\overline{\mathbb{F}}_p$-points of $\mathcal{Z}(\sigma)$ are precisely the representations $\bar{r} : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ having $\sigma \in W^{\mathrm{cris}}(\bar{r})$ (see [BLGG13, Definition 4.1.7] for a precise definition of $W^{\mathrm{cris}}(\bar{r})$).

2.3. **Classification in rank two.** The objective of this section is to classify and describe rank two Breuil-Kisin modules with descent data that satisfy some additional conditions, which we now introduce.

Definition 2.3.1. [CDM18, Definition 3.1.1] *A Breuil-Kisin module $\mathfrak{M}$ defined over an $\mathbb{F}$-algebra R with descent data is said to be of of Hodge type v_0 if it is of rank two, height at most one, and the u-adic valuation of the determinant of each Frobenius map $\Phi_{\mathfrak{M},i} : \varphi^*(\mathfrak{M}_{i-1}) \to \mathfrak{M}_i$ is e.*

Lemma 2.3.2. *Suppose R is a field and $\mathfrak{M}$ is a rank two Breuil-Kisin module over R with tame $\mathbb{F}$-type τ and height at most one. Then Hodge type $\mathbf{v}_0$ condition is equivalent to the strong determinant condition of* [CEGS19, § 3.5].

Proof. We will use [CEGS19, Lem. 3.5.11, Prop. 3.5.12] for the proof. Although these results are stated for coefficients in finite fields, the proofs in fact work for all fields over $\mathbb{F}$.

One direction (strong determinant condition implies Hodge type $\mathbf{v}_0$ condition) follows from [CEGS19, Lem. 3.5.11 (2)]. For the other direction, we observe firstly that $\mathfrak{M}_i/\mathrm{im}(\Phi_{\mathfrak{M},i})$ is a finitely generated torsion $R[\![u]\!]$ module, being a quotient of $\mathfrak{M}_i/u^e\mathfrak{M}_i$. The determinant of $\Phi_{\mathfrak{M},i}$ is the product of the invariants of $\mathfrak{M}_i/\mathrm{im}(\Phi_{\mathfrak{M},i})$ times a unit. Therefore, the sum of u-adic valuations of the invariants is e, implying that the dimension of $\mathfrak{M}_i/\mathrm{im}(\Phi_{\mathfrak{M},i})$ is e. By [CEGS19, Lem. 3.5.11 (1), Lem. 3.5.12], the strong determinant condition is satisfied if and only if $\dim_R\left(\frac{\mathrm{im}(\Phi_{\mathfrak{M},i})}{u^e\mathfrak{M}_i}\right) = e(K/\mathbb{Q}_p)\cdot e(K'/K) = e$. As $\mathfrak{M}_i$ is a rank two free module over $R[\![u]\!]$, $\mathfrak{M}_i/u^e\mathfrak{M}_i$ has dimension $2e$ over R, and thus $\dim_R\left(\frac{\mathrm{im}(\Phi_{\mathfrak{M},i})}{u^e\mathfrak{M}_i}\right) = e$. □

Suppose $\mathfrak{M}$ is Breuil-Kisin module over R satisfying the Hodge type $\mathbf{v}_0$ condition and is of tame principal series $\mathbb{F}$-type $\tau = \eta \oplus \eta'$ with $\eta \neq \eta'$. By Lemma 2.1.4, we know that with respect to inertial bases for $\mathfrak{M}_{i-1}$ and $\mathfrak{M}_i$, the i-th Frobenius map $\Phi_{\mathfrak{M},i}$ is represented by a matrix F_i of the form (2.1.2). Since $\det(F_i) = s_1^{(i)}s_4^{(i)} - vs_2^{(i)}s_3^{(i)}$, the Hodge type $\mathbf{v}_0$ condition implies $v \mid s_1^{(i)}s_4^{(i)}$. This gives us three cases:

(1) If $s_1^{(i)}$ is a non-unit and $s_4^{(i)}$ is a unit mod v, then F_i is of *genre* I_η, denoted by $\mathcal{G}(F_i) = \mathrm{I}_\eta$.
(2) If $s_1^{(i)}$ is a unit and $s_4^{(i)}$ is a non-unit mod v, then F_i is of *genre* $\mathrm{I}_{\eta'}$, denoted by $\mathcal{G}(F_i) = \mathrm{I}_{\eta'}$.
(3) If both $s_1^{(i)}$ and $s_4^{(i)}$ are non-units mod v, then F_i is of *genre* II, denoted by $\mathcal{G}(F_i) = \mathrm{II}$.

A direct calculation shows that if $\{P_i\}_{i\in\mathbb{Z}/f\mathbb{Z}}$ is a set of inertial base change matrices, then $\mathcal{G}(P_i^{-1}\cdot F_i\cdot\varphi(P_{i-1})) = \mathcal{G}(F_i)$. We are therefore justified in defining the *genre of* $\mathfrak{M}_i$ to be $\mathcal{G}(\mathfrak{M}_i) := \mathcal{G}(F_i)$.

Definition 2.3.3. *For a Breuil-Kisin module $\mathfrak{M}$ over R of rank two, height at most one and tame principal series $\mathbb{F}$-type $\tau = \eta\oplus\eta'$, let $\{F_i\}$ be the Frobenius matrices written with respect to a choice of inertial basis for each i. Suppose $\eta\neq\eta'$. We will say that F_i is in η-form if its top left entry is divisible by v. If the bottom right entry is divisible by v, we will say it is in η'-form.*

The property of being in η-form or in η'-form is preserved by inertial base change, and can be seen as a property of the i-th Frobenius map $\Phi_{\mathfrak{M},i}$.

Definition 2.3.4. *A Breuil-Kisin module $\mathfrak{M}$ over an $\mathbb{F}$-algebra R with descent data is regular if it is of Hodge type $\mathbf{v}_0$, tame principal series $\mathbb{F}$-type $\tau = \eta\oplus\eta'$ such that $\eta\neq\eta'$, and with each Frobenius map either in η-form or in η'-form.*

For the rest of this section, our Breuil-Kisin modules will be defined over an $\mathbb{F}$-algebra R, and will be regular. We now turn to show that when R is Artinian local, we can choose inertial bases such that each Frobenius matrix F_i takes one of three forms depending on the genre $\mathcal{G}(\mathfrak{M}_i)$.

Definition 2.3.5. *Let R be an Artinian local ring over $\mathbb{F}$ with maximal ideal $\mathfrak{m}$. Let $\mathfrak{M}$ over R be regular. We say that the Frobenius matrices $\{F_i\}$ (written with respect to an inertial basis for each $\mathfrak{M}_i$) are in CDM form if for $i \in \{1, ..., f-1\}$, we have:*

$$F_i = \begin{cases} \begin{pmatrix} v & 0 \\ A_i u^{\gamma_i} & 1 \end{pmatrix} & \text{if } \mathcal{G}(F_i) = \mathrm{I}_\eta, \\ \begin{pmatrix} 0 & -u^{e-\gamma_i} \\ u^{\gamma_i} & A'_i \end{pmatrix} & \text{if } \mathcal{G}(F_i) = \mathrm{II} \text{ and } F_i \text{ is in } \eta\text{-form}, \\ \begin{pmatrix} 1 & A'_i u^{e-\gamma_i} \\ 0 & v \end{pmatrix} & \text{if } \mathcal{G}(F_i) = \mathrm{I}_{\eta'}, \\ \begin{pmatrix} A_i & -u^{e-\gamma_i} \\ u^{\gamma_i} & 0 \end{pmatrix} & \text{if } \mathcal{G}(F_i) = \mathrm{II} \text{ and } F_i \text{ is in } \eta'\text{-form}, \end{cases}$$

while for $i = 0$,

$$F_0 = \begin{cases} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha' \end{pmatrix} \begin{pmatrix} v & 0 \\ A_0 u^{\gamma_0} & 1 \end{pmatrix} & \text{if } \mathcal{G}(F_0) = \mathrm{I}_\eta, \\ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha' \end{pmatrix} \begin{pmatrix} 0 & -u^{e-\gamma_0} \\ u^{\gamma_0} & A'_0 \end{pmatrix} & \text{if } \mathcal{G}(F_0) = \mathrm{II} \text{ and } F_0 \text{ is in } \eta\text{-form}, \\ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha' \end{pmatrix} \begin{pmatrix} 1 & A'_0 u^{e-\gamma_0} \\ 0 & v \end{pmatrix} & \text{if } \mathcal{G}(F_0) = \mathrm{I}_{\eta'}, \\ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha' \end{pmatrix} \begin{pmatrix} A_0 & -u^{e-\gamma_0} \\ u^{\gamma_0} & 0 \end{pmatrix} & \text{if } \mathcal{G}(F_0) = \mathrm{II} \text{ and } F_0 \text{ is in } \eta'\text{-form}. \end{cases}$$

where each of α, α', A_i, A'_i are elements of R. Moreover, when $\mathcal{G}(F_i) = \mathrm{II}$ and F_i is in η-form, $A'_i \in \mathfrak{m}$. Similarly, when $\mathcal{G}(F_i) = \mathrm{II}$ and F_i is in η'-form, $A_i \in \mathfrak{m}$.

We will describe these matrices in terms of the parameters $(\alpha, \alpha', A_0, A'_0, ..., A_{f-1}, A'_{f-1})$. *If no* A_i *shows up in the description of* F_i, *we set it equal to* 0. *Similarly, if no* A'_i *shows up in the description of* F_i, *we set it equal to* 0. *Note that, in general,* $(\alpha, \alpha', A_0, A'_0, ..., A_{f-1}, A'_{f-1})$ *do not uniquely determine the Frobenius matrices. They do, however, if we know which Frobenius maps are in* η*-form and which are in* η'*-form.*

Definition 2.3.6. *Let* R *be an Artinian local ring over* $\mathbb{F}$ *with maximal ideal* $\mathfrak{m}$. *A regular Breuil-Kisin module* $\mathfrak{M}$ *over* R *is of bad genre if the following conditions are satisfied:*

(1) $\forall i, (\mathcal{G}(F_i), z_i) \in \{(\mathrm{II}, 0), (\mathrm{II}, p-1), (\mathrm{I}_\eta, 1), (\mathrm{I}_\eta, p-1), (\mathrm{I}_{\eta'}, 0), (\mathrm{I}_{\eta'}, p-2)\}$.
(2) *If* $(\mathcal{G}(F_i), z_i) \in \{(\mathrm{II}, 0), (\mathrm{I}_{\eta'}, 0), (\mathrm{I}_{\eta'}, p-2)\}$, *then* $(\mathcal{G}(F_{i+1}), z_{i+1}) \in \{(\mathrm{II}, p-1), (\mathrm{I}_\eta, p-1), (\mathrm{I}_{\eta'}, p-2)\}$.
(3) *If* $(\mathcal{G}(F_i), z_i) \in \{(\mathrm{II}, p-1), (\mathrm{I}_\eta, 1), (\mathrm{I}_\eta, p-1)\}$, *then* $(\mathcal{G}(F_{i+1}), z_{i+1}) \in \{(\mathrm{II}, 0), (\mathrm{I}_\eta, 1), (\mathrm{I}_{\eta'}, 0)\}$.

The $(z_j)_j$ *are as defined in (1.4.1).*

Proposition 2.3.7. *Let* R *be an Artinian local ring over* $\mathbb{F}$ *with maximal ideal* $\mathfrak{m}$. *Let* $\mathfrak{M}$ *be a regular Breuil-Kisin module over* R, *not of bad genre. Then there exists an inertial basis for each* $\mathfrak{M}_i$ *with respect to which the Frobenius matrices are in CDM form (see Definition 2.3.5).*

The proof of Proposition 2.3.7 is very similar to that of [CDM18, Lem. 3.1.20], with slight differences to accommodate Artinian local algebras. Before showing the proof, we first state some of the lemmas and definitions used in the proof.

Definition 2.3.8. *(As in* [CDM18, Lem. 3.1.16]*). Let* R *be an Artinian local ring over* $\mathbb{F}$ *with maximal ideal* $\mathfrak{m}$. *Define an operation* $\mathcal{B}$ *as follows. For any*

$$G = \begin{pmatrix} vs_1 & u^{e-\gamma}s_2 \\ u^\gamma s_3 & s_4 \end{pmatrix}$$

with $s_j \in R[\![v]\!]$ *and* $\det(G) = v\alpha$ *for some* $\alpha \in R[\![v]\!]^*$, *define* $A, A' \in R$ *by* $A \equiv s_3/s_4 \mod v$ *(if* s_4 *is invertible) and* $A' \equiv s_4/s_3 \mod v$ *(if* s_4 *is not invertible). Then* $\mathcal{B}(G)$ *is defined as follows:*

$$\mathcal{B}(G) := \begin{cases} \begin{pmatrix} s_1 - As_2 & u^{e-\gamma}s_2 \\ u^\gamma \frac{s_3 - As_4}{u^e} & s_4 \end{pmatrix} & \text{if } s_4 \in R[\![v]\!]^*, \\ \\ \begin{pmatrix} s_2 - A's_1 & u^{e-\gamma}s_1 \\ u^\gamma \frac{s_4 - A's_3}{u^e} & s_3 \end{pmatrix} & \text{if } s_4 \notin R[\![v]\!]^*. \end{cases}$$

Note that $\det(\mathcal{B}(G)) = \pm\alpha$*, so* $\mathcal{B}(G)$ *is invertible. Furthermore,*

$$G = \begin{cases} \mathcal{B}(G)\begin{pmatrix} v & 0 \\ Au^{\gamma} & 1 \end{pmatrix} & \text{if } s_4 \in R[\![v]\!]^*, \\ \\ \mathcal{B}(G)\begin{pmatrix} 0 & u^{e-\gamma} \\ u^{\gamma} & A' \end{pmatrix} & \text{if } s_4 \notin R[\![v]\!]^*. \end{cases} \tag{2.3.1}$$

For any

$$G = \begin{pmatrix} s_1 & u^{e-\gamma}s_2 \\ u^{\gamma}s_3 & vs_4 \end{pmatrix}$$

with $s_j \in R[\![v]\!]$ *and* $\det(G) = v\alpha$ *for some* $\alpha \in R[\![v]\!]^*$*, define* $A, A' \in R$ *by* $A \equiv s_1/s_2 \mod v$ *(if* s_1 *is not invertible) and* $A' \equiv s_2/s_1 \mod v$ *(if* s_1 *is invertible). Then* $\mathcal{B}(G)$ *is defined in a way that is compatible with the above definitions if we want to interchange* η *and* η'*. That is,*

$$\mathcal{B}(G) := \begin{cases} \begin{pmatrix} s_1 & u^{e-\gamma}\frac{s_2 - A's_1}{u^e} \\ u^{\gamma}s_3 & s_4 - A's_3 \end{pmatrix} & \text{if } s_1 \in R[\![v]\!]^*, \\ \\ \begin{pmatrix} s_2 & u^{e-\gamma}\frac{s_1 - As_2}{u^e} \\ u^{\gamma}s_4 & s_3 - As_4 \end{pmatrix} & \text{if } s_1 \notin R[\![v]\!]^*. \end{cases}$$

Note that $\det(\mathcal{B}(G)) = \pm\alpha$*, so* $\mathcal{B}(G)$ *is invertible. Furthermore,*

$$G = \begin{cases} \mathcal{B}(G)\begin{pmatrix} 1 & A'u^{e-\gamma} \\ 0 & u^e \end{pmatrix} & \text{if } s_1 \in R[\![v]\!]^*, \\ \\ \mathcal{B}(G)\begin{pmatrix} A & u^{e-\gamma} \\ u^{\gamma} & 0 \end{pmatrix} & \text{if } s_1 \notin R[\![v]\!]^*. \end{cases}$$

Lemma 2.3.9. *Consider the matrix*

$$P = \begin{pmatrix} \sigma_1 & u^{e-\gamma_{i-1}}\sigma_2 \\ u^{\gamma_{i-1}}\sigma_3 & \sigma_4 \end{pmatrix},$$

with $\sigma_j \in R[\![v]\!]$ *and det*$(P) \in R[\![u]\!]^*$*. Also let*

$$F = \begin{pmatrix} va & u^{e-\gamma_i}b \\ u^{\gamma_i}c & d \end{pmatrix} \text{ or } \begin{pmatrix} a & u^{e-\gamma_i}b \\ u^{\gamma_i}c & vd \end{pmatrix}$$

with $a, b, c, d \in R[\![v]\!]$ *and* $ad - bc \in R[\![v]\!]^*$*. Further let* M *be the matrix such that* $F = \mathcal{B}(F)M$*, where* $\mathcal{B}$ *is the operation in Definition 2.3.8. Then* $\mathcal{B}(F\varphi(P)) = \mathcal{B}(F)\mathcal{B}(M\varphi(P))$*.*

Proof. Let X be such that $M\varphi(P) = \mathcal{B}(M\varphi(P))X$ and Y be such that $F\varphi(P) = \mathcal{B}(F\varphi(P))Y$. It suffices to show that $X = Y$, because if so, by inverting X and Y in $\mathrm{GL}_2(R(\!(v)\!))$, we can show that $\mathcal{B}(F\varphi(P)) = F\varphi(P)Y^{-1} = \mathcal{B}(F)M\varphi(P)X^{-1} = \mathcal{B}(F)\mathcal{B}(M\varphi(P))$.

We first consider the case where $F = \begin{pmatrix} va & u^{e-\gamma_i}b \\ u^{\gamma_i}c & d \end{pmatrix}$. Note that for any $G = \begin{pmatrix} vs_1 & u^{e-\gamma_i}s_2 \\ u^{\gamma_i}s_3 & s_4 \end{pmatrix}$ with $s_j \in R[\![v]\!]$ and $\det(G) = v\alpha$ for some $\alpha \in R[\![v]\!]^*$, we can calculate $\mathcal{B}(G)$ and scalars A or A' so that (2.3.1) holds. It suffices to show that A and A' do not depend on whether $G = F\varphi(P)$ or $G = M\varphi(P)$. We have

$$F\varphi(P) = \begin{pmatrix} v(a\varphi(\sigma_1) + bu^{ez_i}\varphi(\sigma_3)) & u^{e-\gamma_i}(au^{e(p-z_i)}\varphi(\sigma_2) + b\varphi(\sigma_4)) \\ u^{\gamma_i}(c\varphi(\sigma_1) + du^{ez_i}\varphi(\sigma_3)) & cu^{e(p-z_i)}\varphi(\sigma_2) + d\varphi(\sigma_4) \end{pmatrix}.$$

Let $\overline{x}$ denote the constant part of any $x \in R[\![v]\!]$. By Definition 2.3.8, M is given by:

$$M = \begin{cases} \begin{pmatrix} v & 0 \\ u^{\gamma_i}\frac{\overline{c}}{\overline{d}} & 1 \end{pmatrix} & \text{if } d \in R[\![v]\!]^*, \\ \\ \begin{pmatrix} 0 & u^{e-\gamma_i} \\ u^{\gamma_i} & \frac{\overline{d}}{\overline{c}} \end{pmatrix} & \text{if } d \notin R[\![v]\!]^*. \end{cases}$$

Therefore,

$$M\varphi(P) = \begin{cases} \begin{pmatrix} v\varphi(\sigma_1) & u^{e-\gamma_i}u^{e(p-z_i)}\varphi(\sigma_2) \\ u^{\gamma_i}(\frac{\overline{c}}{\overline{d}}\varphi(\sigma_1) + u^{ez_i}\varphi(\sigma_3)) & \frac{\overline{c}}{\overline{d}}u^{e(p-z_i)}\varphi(\sigma_2) + \varphi(\sigma_4) \end{pmatrix} & \text{if } d \in R[\![v]\!]^*, \\ \\ \begin{pmatrix} vu^{ez_i}\varphi(\sigma_3) & u^{e-\gamma_i}\varphi(\sigma_4) \\ u^{\gamma_i}(\varphi(\sigma_1) + \frac{\overline{d}}{\overline{c}}u^{ez_i}\varphi(\sigma_3)) & u^{e(p-z_i)}\varphi(\sigma_2) + \frac{\overline{d}}{\overline{c}}\varphi(\sigma_4) \end{pmatrix} & \text{if } d \notin R[\![v]\!]^*. \end{cases}$$

Then $X = Y$ follows immediately from Definition 2.3.8. Similar considerations hold for the case $F = \begin{pmatrix} a & u^{e-\gamma_i}b \\ u^{\gamma_i}c & vd \end{pmatrix}$. □

Definition 2.3.10. *Let R be an Artinian local ring over $\mathbb{F}$ with maximal ideal $\mathfrak{m}$. Let $n \in \mathbb{Z}_{\geq 0}$ be the maximum such that $\mathfrak{m}^n \neq 0$. For $t \in \mathbb{Z}_{\geq 0}$, define the ideal $I_t \subset R[\![v]\!]$ as follows:*

$$I_t = \{\sum_{i=\max\{t-n,0\}}^{\infty} a_i v^i \in R[\![v]\!] : a_0 \in \mathfrak{m}^t\}.$$

In other words, for $t \leq n$, $I_t = \{\sum_{i=0}^{\infty} a_i v^i \in R[\![v]\!] : a_0 \in \mathfrak{m}^t\}$. For $t > n$, $I_t = \{\sum_{i=t-n}^{\infty} a_i v^i \in R[\![v]\!]\}$.

Definition 2.3.11. *Let R be an Artinian local ring over $\mathbb{F}$ with maximal ideal $\mathfrak{m}$. We say that P_0 and P_1 are t-close if there exists a matrix*

$$Y = \begin{pmatrix} y_1 & u^{e-\gamma_i}y_2, \\ u^{\gamma_i}y_3 & y_4 \end{pmatrix}$$

satisfying $y_1 \equiv y_2 \equiv y_3 \equiv y_4 \equiv 0 \mod I_t$, *such that* $P_0 = P_1 + Y$.

Lemma 2.3.12. *(c.f.* [CDM18, Lem. 3.1.19]*) Let* R *be an Artinian local ring over* $\mathbb{F}$ *with maximal ideal* $\mathfrak{m}$. *Let*

$$P' = \begin{pmatrix} \sigma'_1 & u^{e-\gamma_{i-1}}\sigma'_2 \\ u^{\gamma_{i-1}}\sigma'_3 & \sigma'_4 \end{pmatrix}$$

be an inertial base change matrix which is t*-close to* Id.

(1) *Let* $M = \begin{pmatrix} v & 0 \\ u^{\gamma_i}a & 1 \end{pmatrix}$ *for* $a \in R$. *Then* $\mathcal{B}(M\varphi(P')) = \begin{pmatrix} \sigma_1 & u^{e-\gamma_i}\sigma_2 \\ u^{\gamma_i}\sigma_3 & \sigma_4 \end{pmatrix}$ *satisfies:*

$$\begin{aligned} &\sigma_1 - 1 \equiv \varphi(\sigma'_1 - 1) \mod I_{t+1}, \\ &\sigma_2 \equiv 0 \mod I_{t+1}, \\ &\sigma_3 \equiv \begin{cases} 0 & \text{if } z_i \neq 1, p-1, \\ \varphi(\sigma'_3) & \text{if } z_i = 1, \\ -a^2\varphi(\sigma'_2) & \text{if } z_i = p-1, \end{cases} \mod I_{t+1}, \\ &\sigma_4 - 1 \equiv \varphi(\sigma'_4 - 1) \mod I_{t+1}. \end{aligned}$$

The congruences also hold true mod v.

(2) *Let* $M = \begin{pmatrix} 1 & u^{e-\gamma_i}a' \\ 0 & v \end{pmatrix}$ *for* $a' \in R$. *Then* $\mathcal{B}(M\varphi(P')) = \begin{pmatrix} \sigma_1 & u^{e-\gamma_i}\sigma_2 \\ u^{\gamma_i}\sigma_3 & \sigma_4 \end{pmatrix}$ *satisfies:*

$$\begin{aligned} &\sigma_1 - 1 \equiv \varphi(\sigma'_1 - 1) \mod I_{t+1}, \\ &\sigma_2 \equiv \begin{cases} 0 & \text{if } z_i \neq 0, p-2, \\ -a'^2\varphi(\sigma'_3) & \text{if } z_i = 0, \\ \varphi(\sigma'_2) & \text{if } z_i = p-2, \end{cases} \mod I_{t+1}, \\ &\sigma_3 \equiv 0 \mod I_{t+1}, \\ &\sigma_4 - 1 \equiv \varphi(\sigma'_4 - 1) \mod I_{t+1}. \end{aligned}$$

The congruences also hold true mod v.

(3) *Let* $M = \begin{pmatrix} 0 & u^{e-\gamma_i} \\ u^{\gamma_i} & a' \end{pmatrix}$ *for* $a' \in \mathfrak{m}$ *or* $M = \begin{pmatrix} a & u^{e-\gamma_i} \\ u^{\gamma_i} & 0 \end{pmatrix}$ *for* $a \in \mathfrak{m}$. *Then* $\mathcal{B}(M\varphi(P')) = \begin{pmatrix} \sigma_1 & u^{e-\gamma_i}\sigma_2 \\ u^{\gamma_i}\sigma_3 & \sigma_4 \end{pmatrix}$ *satisfies:*

$$\begin{aligned} &\sigma_1 - 1 \equiv \varphi(\sigma'_4 - 1) \mod I_{t+1}, \\ &\sigma_2 \equiv \begin{cases} 0 & \text{if } z_i \neq 0, \\ \varphi(\sigma'_3) & \text{if } z_i = 0, \end{cases} \mod I_{t+1}, \end{aligned}$$

$$\sigma_3 \equiv \begin{cases} 0 & \textit{if } z_i \neq p-1, \\ \varphi(\sigma_2') & \textit{if } z_i = p-1, \end{cases} \mod I_{t+1},$$
$$\sigma_4 - 1 \equiv \varphi(\sigma_1' - 1) \mod I_{t+1}.$$

Proof. (1) We have
$$M\varphi(P') = \begin{pmatrix} v\varphi(\sigma_1') & u^{e-\gamma_i}(u^{e(p-z_i)}\varphi(\sigma_2')) \\ u^{\gamma_{i+1}}(u^{ez_i}\varphi(\sigma_3') + a\varphi(\sigma_1')) & \varphi(\sigma_4') + au^{e(p-z_i)}\varphi(\sigma_2') \end{pmatrix}.$$
Let $b \in R$ such that $b = \frac{a\sigma_1' + u^{ez_i}\sigma_3'}{\sigma_4'} \mod v$. Then $\mathcal{B}(M\varphi(P')) = \begin{pmatrix} \sigma_1 & u^{e-\gamma_i}\sigma_2 \\ u^{\gamma_i}\sigma_3 & \sigma_4 \end{pmatrix}$ where
$$\sigma_1 - 1 = \varphi(\sigma_1' - 1) - v^{(p-z_i)}b\varphi(\sigma_2'),$$
$$\sigma_2 = v^{(p-z_i)}\varphi(\sigma_2'),$$
$$v\sigma_3 = a\varphi(\sigma_1') - b\varphi(\sigma_4') + v^{z_i}\varphi(\sigma_3') - abv^{(p-z_i)}\varphi(\sigma_2'),$$
$$\sigma_4 - 1 = av^{(p-z_i)}\varphi(\sigma_2') + \varphi(\sigma_4' - 1).$$
The congruences for σ_1, σ_2 and σ_4 are immediate from the above. For σ_3, consider first the case where $t > n$. Then $\varphi(\sigma_1' - 1) \equiv \varphi(\sigma_4' - 1) \equiv \varphi(\sigma_2') \equiv \varphi(\sigma_3') \equiv 0 \mod I_{t+2}$ (as $p > 2$). Moreover, $a = b$. We have, therefore,
$$\sigma_3 = av^{-1}\left(\varphi(\sigma_1' - 1) - \varphi(\sigma_4' - 1)\right) + v^{z_i - 1}\varphi(\sigma_3') - a^2 v^{(p-1-z_i)}\varphi(\sigma_2').$$
The right hand side is 0 mod I_{t+1} and so are $\varphi(\sigma_3')$ and $a^2\varphi(\sigma_2')$.

Now, let $t \leq n$. First consider the case where $z_i = 0$. The right side of the equality $v\sigma_3 = a\varphi(\sigma_1') - b\varphi(\sigma_4') + v^{z_i}\varphi(\sigma_3') - abv^{(p-z_i)}\varphi(\sigma_2')$ can have no constant terms. Therefore, $v\sigma_3$ depends on $v^p\varphi(\sigma_2')$ and the nonconstant parts of $\varphi(\sigma_1')$, $\varphi(\sigma_3')$ and $\varphi(\sigma_4')$, which are all 0 mod v^2. Therefore, $\sigma_3 \equiv 0 \mod v$ and therefore, mod I_{t+1}. If $z_i \neq 0$, $v\sigma_3$ depends on $v^{p-z_i}\varphi(\sigma_2')$, $v^{z_i}\varphi(\sigma_3')$ and the nonconstant parts of $\varphi(\sigma_1')$ and $\varphi(\sigma_4')$. The latter two terms are 0 mod v^2. This gives us the following equivalence mod v (and hence, mod I_{t+1}):
$$\sigma_3 \equiv v^{z_i - 1}\varphi(\sigma_3') - a^2 v^{p-1-z_i}\varphi(\sigma_2').$$
The desired congruences follow from the same reasoning as for the first case.

(2) We have
$$M\varphi(P') = \begin{pmatrix} \varphi(\sigma_1') + a'u^{e(1+z_i)}\varphi(\sigma_3') & u^{e-\gamma_i}(u^{e(p-1-z_i)}\varphi(\sigma_2') + a'\varphi(\sigma_4')) \\ u^{\gamma_i}(u^{e(1+z_i)}\varphi(\sigma_3')) & v\varphi(\sigma_4') \end{pmatrix}.$$
Let $b' \in R$ such that $b' = \frac{u^{e(p-1-z_i)}\sigma_2' + a'\sigma_4'}{\sigma_1'} \mod v$. Then $\mathcal{B}(M\varphi(P')) = \begin{pmatrix} \sigma_1 & u^{e-\gamma_i}\sigma_2 \\ u^{\gamma_i}\sigma_3 & \sigma_4 \end{pmatrix}$ where
$$\sigma_1 - 1 = \varphi(\sigma_1' - 1) - a'v^{(1+z_i)}\varphi(\sigma_3'),$$
$$v\sigma_2 = v^{(p-1-z_i)}\varphi(\sigma_2') + a'\varphi(\sigma_4') - b'(\varphi(\sigma_1') + a'v^{(1+z_i)}\varphi(\sigma_3')),$$

$$\sigma_3 = v^{(1+z_i)}\varphi(\sigma_3'),$$
$$\sigma_4 - 1 = b'v^{(1+z_i)}\varphi(\sigma_3') + \varphi(\sigma_4' - 1).$$

The desired congruences follow.

(3) Suppose $M = \begin{pmatrix} 0 & u^{e-\gamma_i} \\ u^{\gamma_i} & a' \end{pmatrix}$ for $a' \in \mathfrak{m}$. We have

$$M\varphi(P') = \begin{pmatrix} u^{e(1+z_i)}\varphi(\sigma_3') & u^{e-\gamma_i}\varphi(\sigma_4') \\ u^{\gamma_i}(\varphi(\sigma_1') + a'u^{ez_i}\varphi(\sigma_3')) & u^{e(p-z_i)}\varphi(\sigma_2') + a'\varphi(\sigma_4') \end{pmatrix}.$$

Let $b' \in R$ such that $b' = \frac{a'\sigma_4'}{\sigma_1' + a'v^{z_i}\sigma_3'} \mod v$. Note that $b' \in \mathfrak{m}$ because $a' \in \mathfrak{m}$, and that $b' \equiv a' \mod \mathfrak{m}^2$ because $b' - a' \equiv \frac{a'(\sigma_4' - \sigma_1') - a'^2 v^{z_i}\sigma_3'}{\sigma_1' + a'v^{z_i}\sigma_3'} \equiv 0 \mod (\mathfrak{m}^2, v)$. Then $\mathcal{B}(M\varphi(P')) = \begin{pmatrix} \sigma_1 & u^{e-\gamma_i}\sigma_2 \\ u^{\gamma_i}\sigma_3 & \sigma_4 \end{pmatrix}$, where

$$\sigma_1 - 1 = \varphi(\sigma_4' - 1) - b'v^{z_i}\varphi(\sigma_3'),$$
$$\sigma_2 = v^{z_i}\varphi(\sigma_3'),$$
$$v\sigma_3 = v^{(p-z_i)}\varphi(\sigma_2') + a'\varphi(\sigma_4') - b'(\varphi(\sigma_1') + a'v^{z_i}\varphi(\sigma_3')),$$
$$\sigma_4 - 1 = \varphi(\sigma_1' - 1) + a'v^{z_i}\varphi(\sigma_3').$$

On the other hand, suppose $M = \begin{pmatrix} a & u^{e-\gamma_i} \\ u^{\gamma_i} & 0 \end{pmatrix}$ for $a \in \mathfrak{m}$. Let $b \in R$ such that $b = \frac{a\sigma_1'}{av^{(p-1-z_i)}\sigma_2' + \sigma_4'} \mod v$. Again, note that $b \in \mathfrak{m}$, since $a \in \mathfrak{m}$ and that $b \equiv a \mod \mathfrak{m}^2$. By symmetry (we can interchange η and η' to convert this to a previous computed case), we have $\mathcal{B}(M\varphi(P')) = \begin{pmatrix} \sigma_1 & u^{e-\gamma_i}\sigma_2 \\ u^{\gamma_i}\sigma_3 & \sigma_4 \end{pmatrix}$, where

$$\sigma_1 - 1 = av^{(p-1-z_i)}\varphi(\sigma_2') + \varphi(\sigma_4' - 1),$$
$$v\sigma_2 = v^{(1+z_i)}\varphi(\sigma_3') + a\varphi(\sigma_1') - b(\varphi(\sigma_4') + av^{(p-1-z_i)}\varphi(\sigma_2')),$$
$$\sigma_3 = v^{(p-1-z_i)}\varphi(\sigma_2'),$$
$$\sigma_4 - 1 = \varphi(\sigma_1' - 1) - bv^{(p-1-z_i)}\varphi(\sigma_2').$$

The congruences follow immediately.

□

Proof of Proposition 2.3.7. We set $P_0 = Id$ and construct $P_s = \begin{pmatrix} \sigma_1^{(s)} & u^{e-\gamma_s}\sigma_2^{(s)} \\ u^{\gamma_s}\sigma_3^{(s)} & \sigma_4^{(s)} \end{pmatrix}$ inductively by letting $\mathcal{B}(F_{s+1}\varphi(P_s)) = P_{s+1}\Delta_{s+1}$, where we choose Δ_{s+1} to be a diagonal matrix in $\mathrm{GL}_2(R)$ such that the diagonal entries of P_{s+1} are 1 mod v. Here, the indexing set of the Frobenius matrices F_s is extended to all natural numbers via the natural map $\mathbb{Z} \to \mathbb{Z}/f\mathbb{Z}$.

We let $M_s = \mathcal{B}(F_s)^{-1}F_s$ (so that $M_{s+f} = M_s$). Trivially, P_s and P_{s+f} are 0-close (see Definition 2.3.11). Suppose P_s and P_{s+f} are t-close for $t \geq 0$. Let $Y = \begin{pmatrix} y_1 & u^{e-\gamma_s}y_2 \\ u^{\gamma_s}y_3 & y_4 \end{pmatrix}$ be such that $P_s = P_{s+f} + Y$.

We use Lemma 2.3.12 to calculate the differences between $\mathcal{B}(M_{s+1}\varphi(P_s))$ and $\mathcal{B}(M_{s+f+1}\varphi(P_{s+f}))$ mod I_{t+1}. Let $\mathcal{B}(M_{s+1}\varphi(P_s)) = \mathcal{B}(M_{s+f+1}\varphi(P_{s+f})) + Y'$ where $Y' = \begin{pmatrix} y_1' & u^{e-\gamma_{s+1}}y_2' \\ u^{\gamma_{s+1}}y_3' & y_4' \end{pmatrix}$. Since the diagonal entries of both P_s and P_{s+f} equal 1 mod v, y_1 and y_4 are 0 mod v and therefore, y_1' and y_4' must be 0 mod I_{t+1}. Moreover, at least one of y_2' and y_3' is 0 mod I_{t+1}. Either (but not both) of y_2' and y_3' can depend on either (but not both) of y_2 and y_3 mod I_{t+1} depending on the genre of F_s and the value of z_{s+1}. Since y_2 and y_3 are 0 mod I_t, the same ends up being true for y_2' and y_3'.

Using Lemma 2.3.9, we have:

$$\begin{aligned}
&\mathcal{B}(F_{s+f+1}\varphi(P_{s+f}))^{-1}\mathcal{B}(F_{s+1}\varphi(P_s)) \\
&= \mathcal{B}(M_{s+1}\varphi(P_{s+f}))^{-1}\mathcal{B}(F_{s+1})^{-1}\mathcal{B}(F_{s+1})\mathcal{B}(M_{s+1}\varphi(P_s)) \\
&= \mathcal{B}(M_{s+1}\varphi(P_{s+f}))^{-1}(\mathcal{B}(M_{s+1}\varphi(P_{s+f}) + Y') \\
&= Id + \mathcal{B}(M_{s+1}\varphi(P_{s+f}))^{-1}Y'.
\end{aligned}$$

Since P_{s+f} is 0-close to Id and the diagonal terms of P_{s+f} are 1 mod v , we can use the congruences in Lemma 2.3.12 to conclude that $\mathcal{B}(M_{s+1}\varphi(P_{s+f}))^{-1}$ has the form $Id + \begin{pmatrix} x_1 & u^{e-\gamma_{s+1}}x_2 \\ u^{\gamma_{s+1}}x_3 & x_4 \end{pmatrix}$, with $x_1, x_4 \in I_1$. Therefore,

$$\mathcal{B}(M_{s+1}\varphi(P_{s+f}))^{-1}Y' = Y' + \begin{pmatrix} x_1y_1' + vx_2y_3' & u^{e-\gamma_{s+1}}(x_1y_2' + x_2y_4') \\ u^{\gamma_{s+1}}(x_3y_1' + x_4y_3') & vx_3y_2' + x_4y_4' \end{pmatrix}$$

is $(t+1)$-close to Y'. This is evident when $t \leq n$, because in that case, $I_1I_t \subset I_{t+1}$. On the other hand, when $t > n$, y_2' and y_3' are already 0 mod I_{t+1}, and the assertion follows.

This implies that $\mathcal{B}(F_{s+f+1}\varphi(P_{s+f}))$ and $\mathcal{B}(F_{s+1}\varphi(P_s))$ have the same diagonal entries mod I_{t+1} and consequently, $\Delta_{s+1} \equiv \Delta_{s+f+1} \mod \mathfrak{m}^{t+1}$. Further,

$$\begin{aligned}
P_{s+f+1}^{-1}P_{s+1} &= \Delta_{s+f+1}\mathcal{B}(F_{s+f+1}\varphi(P_{s+f}))^{-1}\mathcal{B}(F_{s+1}\varphi(P_s))\Delta_{s+1}^{-1} \\
&= Id + \Delta_{s+f+1}\mathcal{B}(M_{s+1}\varphi(P_{s+f}))^{-1}Y'\Delta_{s+1}^{-1},
\end{aligned}$$

where the entries of $P_{s+f+1}^{-1}P_{s+1} - Id = \Delta_{s+f+1}\mathcal{B}(M_{s+1}\varphi(P_{s+f}))^{-1}Y'\Delta_{s+1}^{-1}$ differ from those of $\mathcal{B}(M_{s+1}\varphi(P_{s+f}))^{-1}Y'$ by some scalars congruent to 1 mod I_{t+1}. As a result, $P_{s+f+1}^{-1}P_{s+1} - Id$ is $(t+1)$-close to Y'.

Note that since the diagonal terms of P_{s+f+1} are 1 mod v, $Y'' = P_{s+f+1}(P_{s+f+1}^{-1}P_{s+1} - Id)$ is $(t+1)$-close to $P_{s+f+1}^{-1}P_{s+1} - Id$. Therefore, $P_{s+1} = P_{s+f+1} + Y''$ where Y'' is $(t+1)$-close to Y'.

We now induct on s. It is evident from the pattern of dependencies in Lemma 2.3.12 that if $\mathfrak{M}$ is not of bad genre (see Definition 2.3.6), the chain of dependencies on y_2 and y_3 is broken during the induction process and we get a Cauchy sequence $(P_{s+nf})_n$. Therefore, we can set $P^{(i)} = \lim_{n\to\infty} P_{i+nf}$ and let $F_i' := (P^{(i)})^{-1}F_i\varphi(P^{(i-1)})$. Then F_i' has the following form:

$$F_i' = \begin{cases} \Delta_i \begin{pmatrix} v & 0 \\ A_i u^{\gamma_i} & 1 \end{pmatrix} & \text{if } \mathcal{G}(F_i) = \mathrm{I}_\eta, \\ \Delta_i \begin{pmatrix} 0 & u^{e-\gamma_i} \\ u^{\gamma_i} & A_i' \end{pmatrix} & \text{if } \mathcal{G}(F_i) = \mathrm{II} \text{ and } F_i \text{ is in } \eta\text{-form}, \\ \Delta_i \begin{pmatrix} 1 & A_i' u^{e-\gamma_i} \\ 0 & v \end{pmatrix} & \text{if } \mathcal{G}(F_i) = \mathrm{I}_{\eta'}, \\ \Delta_i \begin{pmatrix} A_i & u^{e-\gamma_i} \\ u^{\gamma_{i+1}} & 0 \end{pmatrix} & \text{if } \mathcal{G}(F_i) = \mathrm{II} \text{ and } F_i \text{ is in } \eta'\text{-form}, \end{cases} \tag{2.3.2}$$

where Δ_i are diagonal matrices with entries in R^* and $A_i, A_i' \in R$. Note that when $\mathcal{G}(F_i) = \mathrm{II}$, the diagonal terms of F_i' are in $\mathfrak{m}$.

Now we do one final base change by diagonal scalar matrices Q_i where $Q_0 = Id$ and $Q_1, ..., Q_{f-1}$ are defined inductively so that $G_i = Q_i^{-1} F_i Q_{i-1}$ is in CDM form. □

Remark 2.3.13. *Note that the definition of bad genre in Definition 2.3.6 is minimal in the sense that for any choice of bad genre, it is easy to write Frobenius matrices that prohibit the convergence of the algorithm in the proof of Proposition 2.3.7.*

2.4. **Base changes.** We continue to assume that our Breuil-Kisin modules are regular (see Definition 2.3.4).

Proposition 2.4.1. [CDM18, Prop. 3.1.22] *Let R be an Artinian local ring over $\mathbb{F}$ with maximal ideal $\mathfrak{m}$. Let $\mathfrak{M}$ be a regular Breuil-Kisin module over R, not of bad genre. Let the Frobenius matrices G_i be in the CDM form with parameters $(\alpha, \alpha', A_0, A_0', ..., A_{f-1}, A_{f-1}')$. Suppose there exist inertial base change matrices P_i, so that if $F_i = P_i^{-1} G_i \varphi(P_{i-1})$, then $\{F_i\}_i$ are also in the CDM form with some parameters $(\beta, \beta', B_0, B_0', ..., B_{f-1}, B_{f-1}')$. Then the following hold true:*

(1) *For all i, P_i are necessarily of the form:*
$$P_i = \begin{pmatrix} \lambda_i & 0 \\ 0 & \mu_i \end{pmatrix} \in \mathrm{GL}_2(R).$$
(2) *If $\mathcal{G}(F_i) \in \{\mathrm{I}_\eta, \mathrm{I}_{\eta'}\}$, then for $i \in \{1, \dots, f-1\}$, $\lambda_i = \lambda_{i-1}$ and $\mu_i = \mu_{i-1}$.*
(3) *If $\mathcal{G}(F_i) = \mathrm{II}$, then for $i \in \{1, \dots, f-1\}$, $\lambda_i = \mu_{i-1}$ and $\mu_i = \lambda_{i-1}$.*

Therefore the parameters $(\beta, \beta', B_0, B_0', ..., B_{f-1}, B_{f-1}')$ are obtained by suitably scaling the parameters $(\alpha, \alpha', A_0, A_0', ..., A_{f-1}, A_{f-1}')$ as dictated by the base change matrices.

The proof of this proposition uses a few more lemmas given below.

Lemma 2.4.2. *Assume the setting of Proposition 2.4.1. Let* $C_i = \mathcal{B}(G_i\varphi(P_{i-1}))^{-1}G_i\varphi(P_{i-1})$. *Then* $C_i = \Delta_i^{-1}F_i$ *where* Δ_i *is the identity matrix for* $i \in \{1, \ldots, f-1\}$ *and equals* $\left(\begin{smallmatrix} \beta & 0 \\ 0 & \beta' \end{smallmatrix}\right)$*, for* $i = 0$.

Proof. Let c_i, c_i' be such that:

$$C_i = \begin{cases} \begin{pmatrix} v & 0 \\ c_i u^{\gamma_i} & 1 \end{pmatrix} & \text{if } \mathcal{G}(G_i) = \mathrm{I}_\eta, \\ \\ \begin{pmatrix} 0 & u^{e-\gamma_i} \\ u^{\gamma_i} & c_i' \end{pmatrix} & \text{if } \mathcal{G}(G_i) = \mathrm{II} \text{ and } G_i \text{ is in } \eta\text{-form}, \\ \\ \begin{pmatrix} 1 & c_i' u^{e-\gamma_i} \\ 0 & v \end{pmatrix} & \text{if } \mathcal{G}(G_i) = \mathrm{I}_{\eta'}, \\ \\ \begin{pmatrix} c_i & u^{e-\gamma_i} \\ u^{\gamma_i} & 0 \end{pmatrix} & \text{if } \mathcal{G}(G_i) = \mathrm{II} \text{ and } G_i \text{ is in } \eta'\text{-form}. \end{cases} \tag{2.4.1}$$

Since $P_i^{-1}G_i\varphi(P_{i-1}) = F_i$, we have $P_i^{-1}\mathcal{B}(G_i\varphi(P_{i-1}))C_i = F_i$. Inverting C_i in $\mathrm{GL}_2(R(\!(u)\!))$, we obtain that $F_iC_i^{-1} = P_i^{-1}\mathcal{B}(G_i\varphi(P_{i-1}))$. Notice that $P_i^{-1}\mathcal{B}(G_i\varphi(P_{i-1}))$ is in $\mathrm{GL}_2(R[\![u]\!])$ and therefore all the entries of $F_iC_i^{-1}$ must be in $R[\![u]\!]$.

Now, consider the case where $\mathcal{G}(G_i) = \mathrm{I}_\eta$.

$$\begin{aligned} F_iC_i^{-1} &= \Delta_i \begin{pmatrix} v & 0 \\ B_i u^{\gamma_i} & 1 \end{pmatrix} \begin{pmatrix} v^{-1} & 0 \\ -c_i u^{\gamma_i - e} & 1 \end{pmatrix} \\ &= \Delta_i \begin{pmatrix} 1 & 0 \\ (B_i - c_i)u^{-e+\gamma_i} & 1 \end{pmatrix} \end{aligned}$$

We conclude that the entries of $F_iC_i^{-1}$ are in $R[\![u]\!]$ if and only if $c_i = B_i$ or in other words, $C_i = \Delta_i^{-1}F_i$. The other three cases involve similar computations and conclusions, and are omitted. □

Lemma 2.4.3. *Assume the setting of Proposition 2.4.1. If* $i \in \{1, \ldots, f-1\}$*, then* $P_i = \mathcal{B}(G_i\varphi(P_{i-1}))$*. Furthermore,* $P_0 = \mathcal{B}(G_0\varphi(P_{f-1}))\left(\begin{smallmatrix} \beta^{-1} & 0 \\ 0 & \beta'^{-1} \end{smallmatrix}\right)$.

Proof. By Lemma 2.4.2, $P_i^{-1}\mathcal{B}(G_i\varphi(P_{i-1}))\Delta_i^{-1}F_i = P_i^{-1}G_i\varphi(P_{i-1}) = F_i$. Inverting F_i in $\mathrm{GL}_2(R(\!(u)\!))$, we have $P_i^{-1}\mathcal{B}(G_i\varphi(P_{i-1}))\Delta_i^{-1} = Id$, and therefore, $P_i = \mathcal{B}(G_i\varphi(P_{i-1}))\Delta_i^{-1}$. □

Lemma 2.4.4. *Assume the setting of Proposition 2.4.1. Assume both the diagonal entries of P_0 equal* 1 *mod v. Then $P_i = Id$ for all i.*

Proof. Suppose that P_0 is t-close to Id (this is automatically true for $t = 0$ from the hypothesis in the statement of the Lemma). We apply Lemma 2.3.12 successively to compute the congruences for $P_i = \mathcal{B}(G_i\varphi(P_{i-1}))$ as i goes from 1 to $f-1$, and then finally for $\mathcal{B}(\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha'^{-1} \end{pmatrix} G_0\varphi(P_{f-1}))$.

We obtain that

$$\mathcal{B}(\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha'^{-1} \end{pmatrix} G_0\varphi(P_{f-1})) = \begin{pmatrix} \sigma_1 & u^{e-\gamma_0}\sigma_2 \\ u^{\gamma_0}\sigma_3 & \sigma_4 \end{pmatrix},$$

where

$$\sigma_1 - 1 \equiv \sigma_2 \equiv \sigma_3 \equiv \sigma_4 - 1 \equiv 0 \mod I_{t+1}.$$

By Lemma 2.4.3,

$$\begin{aligned} P_0 &= \mathcal{B}(G_0\varphi(P_{f-1})) \begin{pmatrix} \beta^{-1} & 0 \\ 0 & \beta'^{-1} \end{pmatrix} \\ &= \begin{pmatrix} \alpha & 0 \\ 0 & \alpha' \end{pmatrix} \mathcal{B}(\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha'^{-1} \end{pmatrix} G_0\varphi(P_{f-1})) \begin{pmatrix} \beta^{-1} & 0 \\ 0 & \beta'^{-1} \end{pmatrix} \qquad \text{(using Lemma 2.3.9)} \\ &= \begin{pmatrix} \alpha & 0 \\ 0 & \alpha' \end{pmatrix} \begin{pmatrix} \sigma_1 & u^{e-\gamma_0}\sigma_2 \\ u^{\gamma_0}\sigma_3 & \sigma_4 \end{pmatrix} \begin{pmatrix} \beta^{-1} & 0 \\ 0 & \beta'^{-1} \end{pmatrix}. \end{aligned}$$

Recalling that P_0 has diagonal entries equal to 1 mod v, we have:

$$\begin{aligned} &\alpha\beta^{-1}\sigma_1 - 1 \equiv \alpha'\beta'^{-1}\sigma_4 - 1 \equiv 0 \mod v, \\ &\alpha\beta^{-1}\sigma_1 - 1 \equiv \alpha'\beta'^{-1}\sigma_4 - 1 \equiv 0 \mod I_{t+1}, \\ &\alpha\beta'^{-1}\sigma_2 \equiv \alpha'\beta^{-1}\sigma_3 \equiv 0 \mod I_{t+1}. \end{aligned}$$

The mod v congruence shows that $\alpha = \beta$ and $\alpha' = \beta'$. Therefore, P_0 is $(t+1)$-close to Id. Induction on t gives us the desired proof. □

Proof of Proposition 2.4.1. Suppose the top left entry of P_0 is λ_0 mod v, while the bottom right entry is μ_0 mod v, where $\lambda_0, \mu_0 \in R^*$. Let $Q_i := \begin{pmatrix} \lambda_i^{-1} & 0 \\ 0 & \mu_i^{-1} \end{pmatrix}$ where λ_i and μ_i are defined in the following manner for $i \in \{1, \dots, f-1\}$: If $\mathcal{G}(G_i) \in \{\mathrm{I}_\eta, \mathrm{I}_{\eta'}\}$, then we let $\lambda_i = \lambda_{i-1}$ and $\mu_i = \mu_{i-1}$. If $\mathcal{G}(G_i) = \mathrm{II}$, we let $\lambda_i = \mu_{i-1}$ and $\mu_i = \lambda_{i-1}$. To prove the proposition, we must show that $P_i = Q_i^{-1}$.

Observe that the matrices $H_i = Q_i^{-1}F_i\varphi(Q_{i-1})$ are still in CDM form (see Definition 2.3.5). We now consider the base change given by the matrices P_iQ_i, that transforms G_i to H_i. By the choice of λ_0 and μ_0, the diagonal entries of P_0Q_0 equal 1 mod v. Applying Lemma 2.4.4, we have $P_iQ_i = Id$ for all i, and therefore $P_i = Q_i^{-1}$. □

Corollary 2.4.5. *Let R be an Artinian local $\mathbb{F}$-algebra. Let $\mathfrak{M}$ be a regular Breuil-Kisin module defined over R and not of bad genre. Let $\{F_i\}_i$ and $\{G_i\}_i$ be two sets of Frobenius matrices for $\mathfrak{M}$ written with respect to different sets of inertial bases. Then the base change matrices $\{P_i\}_i$ to go from $\{F_i\}_i$ to $\{G_i\}_i$ are unique up to multiplying each of the P_i by a fixed scalar matrix.*

Proof. Since each set of Frobenius matrices can be transformed into CDM form, it suffices to check the assertion when $\{F_i\}_i$ and $\{G_i\}_i$ are assumed to be in CDM form. From the way the parameters for the Frobenius matrices transform under base change, it is immediate that the base change matrices are uniquely determined up to scalar multiples. □

For the remainder of this section, we will make the following assumption for a Breuil-Kisin module $\mathfrak{M}$ defined over R.

Assumption 2.4.6. *$\mathfrak{M}$ is a regular Breuil-Kisin module over R, not of bad genre. Each of its Frobenius maps is in η-form, and none are in η'-form.*

The assumption is justified because allowing some Frobenius matrices to be in η'-form will offer very little advantage in our eventual conclusions but inundate the text with significantly more notation - a discussion of the effect of allowing some Frobenius matrices to be in η'-form is in the Appendix.

Via Proposition 2.3.7, we can now describe Frobenius maps very parsimoniously using matrices in CDM form. Base changes between CDM forms also have an easy description using Proposition 2.4.1. This bring us one step closer to finding a finite presentation of the stack of Breuil-Kisin modules. We now turn our attention to furthering this process, specifically to understanding the base changes that allowed us to write the Frobenius matrices in the CDM form. Specifically, we will be studying the matrices $P^{(i)} = \lim_{n\to\infty} P_{i+nf}$ showing up in the proof of Proposition 2.3.7. We will also analyze obstructions to a parsimonious description, one of which we have already seen show up as a 'bad genre' condition. We have seen that $\mathfrak{M}$ can be of bad genre only if the infinite sequence $(z_i)_{i\in\mathbb{Z}}$ is made up entirely of the building blocks 1 and $(0, p-1)$. On the other hand, if $(z_i)_{i\in\mathbb{Z}}$ is such, we can find an $\mathfrak{M}$ of bad genre by choosing the entries of the Frobenius matrices suitably. This motivates the following definition.

Definition 2.4.7. *We say that a tame principal series $\mathbb{F}$-type τ faces the first obstruction if $(z_i)_{i\in\mathbb{Z}}$ is made up entirely of the building blocks 1 and $(0, p-1)$.*

Proposition 2.4.8. *Let R be an Artinian local ring over $\mathbb{F}$ with maximal ideal $\mathfrak{m}$. Let $\mathfrak{M}$ be a Breuil-Kisin module over R satisfying Assumption 2.4.6. Suppose with respect to an inertial basis, F_i has the form*

$$\begin{pmatrix} va_i & u^{e-\gamma_i}b_i \\ u^{\gamma_i}c_i & d_i \end{pmatrix}$$

with $a_i, b_i, c_i, d_i \in R$. Let $P^{(j)} = \lim_{n\to\infty} P_{j+nf}$ denote the base change matrices described in the proof of Proposition 2.3.7. Let $F_i' = (P^{(i)})^{-1}F_i\varphi(P_{i-1})$ be the matrix in (2.3.2), and explicitly, let

$$F_i' = \begin{pmatrix} va_i' & b_i'u^{e-\gamma_i} \\ c_i'u^{\gamma_i} & d_i' \end{pmatrix}.$$

Define a left action of upper unipotent matrices on η-form Frobenius matrices in the following manner:

$$\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \star \begin{pmatrix} va_i & u^{e-\gamma_i}b_i \\ u^{\gamma_i}c_i & d_i \end{pmatrix} = \begin{pmatrix} v(a_i+yc_i) & u^{e-\gamma_i}(b_i+yd_i) \\ u^{\gamma_i}c_i & d_i \end{pmatrix}.$$

The following statements are true:

(1) *Suppose $(z_i)_{i\in\mathbb{Z}}$ does not contain the subsequence $(p-1,1,...,1,0)$ (where the number of 1's is allowed to be zero). Then there exists an upper unipotent U_i for each i satisfying $U_i \star F_i = F_i'$.*
(2) *If $(z_i)_i$ contains the subsequence $(p-1,1,...,1,0)$, then there exists a set of Frobenius matrices $\{F_i\}$ such that, for some l, no unipotent matrix U satisfies $U \star F_l = F_l'$.*

The proof will use the following lemma.

Lemma 2.4.9. *Consider the setup of Proposition 2.4.8. Suppose that the base change matrices $P^{(j)}$ are given by*

$$\begin{pmatrix} q_j & u^{e-\gamma_j}r_j \\ u^{\gamma_j}s_j & t_j \end{pmatrix}.$$

For any $\sigma \in R[\![v]\!]$, denote by $\overline{\sigma}$ the constant part of σ. Then

$$F_i' = \begin{cases} Ad\begin{pmatrix} 1 & 0 \\ 0 & \frac{c_i+d_i\overline{s_{i-1}}}{c_i} \end{pmatrix}\left(\begin{pmatrix} (a_i-\frac{c_i}{d_i}b_i)v & 0 \\ u^{\gamma_i}c_i & d_i \end{pmatrix}\right) & \text{if } \mathcal{G}(F_i)=\mathrm{I}_\eta, z_i=0, \\ \begin{pmatrix} (a_i-\frac{c_i}{d_i}b_i)v & 0 \\ u^{\gamma_i}c_i & d_i \end{pmatrix} & \text{if } \mathcal{G}(F_i)=\mathrm{I}_\eta, z_i\neq 0, \\ Ad\begin{pmatrix} 1 & 0 \\ 0 & \frac{c_i+d_i\overline{s_{i-1}}}{c_i} \end{pmatrix}\left(\begin{pmatrix} 0 & u^{e-\gamma_i}(b_i-\frac{d_i}{c_i}a_i) \\ u^{\gamma_i}c_i & d_i \end{pmatrix}\right) & \text{if } \mathcal{G}(F_i)=\mathrm{II}, z_i=0, \\ \begin{pmatrix} 0 & u^{e-\gamma_i}(b_i-\frac{d_i}{c_i}a_i) \\ u^{\gamma_i}c_i & d_i \end{pmatrix} & \text{if } \mathcal{G}(F_i)=\mathrm{II}, z_i\neq 0, \end{cases}$$

where $Ad\ M\ (N)$ denotes the matrix MNM^{-1}.

Proof. Using the definition of the operator $\mathcal{B}$ in Definition 2.3.8 and our calculations in Lemma 2.3.9, we have

$$\mathcal{B}(F_i)^{-1}F_i = M_i = \begin{cases} \begin{pmatrix} v & 0 \\ u^{\gamma_i}\frac{c_i}{d_i} & 1 \end{pmatrix} & \text{if } \mathcal{G}(F_i) = \mathrm{I}_\eta, \\ \\ \begin{pmatrix} 0 & u^{e-\gamma_i} \\ u^{\gamma_i} & \frac{d_i}{c_i} \end{pmatrix} & \text{if } \mathcal{G}(F_i) = \mathrm{II}, \end{cases}$$

$$\mathcal{B}(F_i) = \begin{cases} \begin{pmatrix} a_i - \frac{c_i}{d_i}b_i & u^{e-\gamma_i}b_i \\ 0 & d_i \end{pmatrix} & \text{if } \mathcal{G}(F_i) = \mathrm{I}_\eta, \\ \\ \begin{pmatrix} b_i - \frac{d_i}{c_i}a_i & u^{e-\gamma_i}a_i \\ 0 & c_i \end{pmatrix} & \text{if } \mathcal{G}(F_i) = \mathrm{II}, \end{cases}$$

and

$$\mathcal{B}(M_i\varphi(P^{(i-1)}))^{-1}M_i\varphi(P^{(j)}) = \begin{cases} \begin{pmatrix} v & 0 \\ u^{\gamma_i}(\frac{c_i}{d_i} + \overline{s_{i-1}}) & 1 \end{pmatrix} & \text{if } \mathcal{G}(F_i) = \mathrm{I}_\eta, z_i = 0, \\ \\ \begin{pmatrix} v & 0 \\ u^{\gamma_i}\frac{c_i}{d_i} & 1 \end{pmatrix} & \text{if } \mathcal{G}(F_i) = \mathrm{I}_\eta, z_i \neq 0, \\ \\ \begin{pmatrix} 0 & u^{e-\gamma_i} \\ u^{\gamma_i} & \frac{\frac{d_i}{c_i}}{1+\frac{d_i}{c_i}\overline{s_{i-1}}} \end{pmatrix} & \text{if } \mathcal{G}(F_i) = \mathrm{II}, z_i = 0, \\ \\ \begin{pmatrix} 0 & u^{e-\gamma_i} \\ u^{\gamma_i} & \frac{d_i}{c_i} \end{pmatrix} & \text{if } \mathcal{G}(F_i) = \mathrm{II}, z_i \neq 0. \end{cases}$$

Recall that by Lemma 2.3.9, $\mathcal{B}(F_i\varphi(P^{(i-1)})) = \mathcal{B}(F_i)\mathcal{B}(M_i\varphi(P^{(i-1)}))$ and by the calculations in Lemma 2.3.12, $\mathcal{B}(M_i\varphi(P^{(i-1)}))$ is Id mod u if $\mathcal{G}(F_i) \neq \mathrm{II}$ or if $\mathcal{G}(F_i) = \mathrm{II}$ but $z_i \neq 0$. By the algorithm in the proof of Proposition 2.3.7, we find that $\mathcal{B}(F_i\varphi(P^{(i-1)})) = P^{(i)}\Delta^{(i)}$ for a suitable diagonal scalar matrix $\Delta^{(i)}$ chosen such that the diagonal entries of $P^{(i)}$ are 1 mod v or, in other words, such that $P^{(i)}$ is Id mod u. Therefore if $\mathcal{G}(F_i) \neq \mathrm{II}$ or if $\mathcal{G}(F_i) = \mathrm{II}$ but

$z_i \neq 0$, $\Delta^{(i)} \equiv \mathcal{B}(F_i) \mod u$. If $\mathcal{G}(F_i) = \mathrm{II}$ and $z_i = 0$, Lemma 2.3.12 gives us the following equivalence mod u:

$$\begin{aligned}\Delta^{(i)} &\equiv \mathcal{B}(F_i)\begin{pmatrix} 1 - \frac{\frac{d_i}{c_i}\overline{s_{i-1}}}{1+\frac{d_i}{c_i}\overline{s_{i-1}}} & 0 \\ 0 & 1 + \frac{d_i}{c_i}\overline{s_{i-1}} \end{pmatrix} \\ &\equiv \mathcal{B}(F_i)\begin{pmatrix} \frac{c_i}{c_i + d_i\overline{s_{i-1}}} & 0 \\ 0 & \frac{c_i + d_i\overline{s_{i-1}}}{c_i} \end{pmatrix}.\end{aligned}$$

Letting $D_i = a_i d_i - b_i c_i$,

$$\Delta^{(i)} = \begin{cases} \begin{pmatrix} \frac{D_i}{d_i} & 0 \\ 0 & d_i \end{pmatrix} & \text{if } \mathcal{G}(F_i) = \mathrm{I}_\eta, \\ \begin{pmatrix} \frac{-D_i}{c_i + d_i\overline{s_{i-1}}} & 0 \\ 0 & c_i + d_i\overline{s_{i-1}} \end{pmatrix} & \text{if } \mathcal{G}(F_i) = \mathrm{II}, z_i = 0, \\ \begin{pmatrix} \frac{-D_i}{c_i} & 0 \\ 0 & c_i \end{pmatrix} & \text{if } \mathcal{G}(F_i) = \mathrm{II}, z_i \neq 0. \end{cases}$$

Now we compute F_i':

$$\begin{aligned} F_i' &= (P^{(i)})^{-1} F_i \varphi(P^{(i-1)}) \\ &= \Delta^{(i)} \mathcal{B}(F_i\varphi(P^{(i-1)}))^{-1} F_i \varphi(P^{(i-1)}) \\ &= \Delta^{(i)} \mathcal{B}(M_i\varphi(P^{(i-1)}))^{-1} M_i \varphi(P^{(i-1)}) \\ &= \begin{cases} \begin{pmatrix} D_i/d_i & 0 \\ 0 & d_i \end{pmatrix} \begin{pmatrix} v & 0 \\ u^{\gamma_i}(\frac{c_i + d_i\overline{s_{i-1}}}{d_i}) & 1 \end{pmatrix} & \text{if } \mathcal{G}(F_i) = \mathrm{I}_\eta, z_i = 0, \\ \begin{pmatrix} D_i/d_i & 0 \\ 0 & d_i \end{pmatrix} \begin{pmatrix} v & 0 \\ u^{\gamma_i}\frac{c_i}{d_i} & 1 \end{pmatrix} & \text{if } \mathcal{G}(F_i) = \mathrm{I}_\eta, z_i \neq 0, \\ \begin{pmatrix} \frac{-D_i}{c_i + d_i\overline{s_{i-1}}} & 0 \\ 0 & c_i + d_i\overline{s_{i-1}} \end{pmatrix} \begin{pmatrix} 0 & u^{e-\gamma_i} \\ u^{\gamma_i} & \frac{d_i}{c_i + d_i\overline{s_{i-1}}} \end{pmatrix} & \text{if } \mathcal{G}(F_i) = \mathrm{II}, z_i = 0, \\ \begin{pmatrix} -D_i/c_i & 0 \\ 0 & c_i \end{pmatrix} \begin{pmatrix} 0 & u^{e-\gamma_i} \\ u^{\gamma_i} & \frac{d_i}{c_i} \end{pmatrix} & \text{if } \mathcal{G}(F_i) = \mathrm{II}, z_i \neq 0, \end{cases} \end{aligned}$$

$$= \begin{cases} Ad\begin{pmatrix} 1 & 0 \\ 0 & \frac{c_i + d_i \overline{s_{i-1}}}{c_i} \end{pmatrix}\left(\begin{pmatrix} (a_i - \frac{c_i}{d_i} b_i)v & 0 \\ u^{\gamma_i} c_i & d_i \end{pmatrix}\right) & \text{if } \mathcal{G}(F_i) = \mathrm{I}_\eta, z_i = 0, \\ \begin{pmatrix} (a_i - \frac{c_i}{d_i} b_i)v & 0 \\ u^{\gamma_i} c_i & d_i \end{pmatrix} & \text{if } \mathcal{G}(F_i) = \mathrm{I}_\eta, z_i \neq 0, \\ Ad\begin{pmatrix} 1 & 0 \\ 0 & \frac{c_i + d_i \overline{s_{i-1}}}{c_i} \end{pmatrix}\left(\begin{pmatrix} 0 & u^{e-\gamma_i}(b_i - \frac{d_i}{c_i} a_i) \\ u^{\gamma_i} c_i & d_i \end{pmatrix}\right) & \text{if } \mathcal{G}(F_i) = \mathrm{II}, z_i = 0, \\ \begin{pmatrix} 0 & u^{e-\gamma_i}(b_i - \frac{d_i}{c_i} a_i) \\ u^{\gamma_i} c_i & d_i \end{pmatrix} & \text{if } \mathcal{G}(F_i) = \mathrm{II}, z_i \neq 0. \end{cases}$$

□

Proof of Proposition 2.4.8. By Lemma 2.4.9, F_i' can be obtained via left unipotent action whenever $z_i \neq 0$. If $z_i = 0$, then F_i' can be obtained via left unipotent action if and only if $s_i \equiv 0 \mod v$.

Now, suppose $z_i = 0$ and $s_{i-1} \not\equiv 0 \mod v$. Recall that $P^{(i-1)} = \mathcal{B}(F_{i-1}\varphi(P^{(i-2)}))(\Delta^{i-1})^{-1} = \mathcal{B}(F_{i-1})\mathcal{B}(M_{i-1}\varphi(P^{(i-2)}))(\Delta^{i-1})^{-1}$.

By the explicit calculations in Lemma 2.4.9, $\mathcal{B}(F_{i-1})$ is upper triangular. Therefore, $s_{i-1} \not\equiv 0$ if and only if $\mathcal{B}(M_{i-1}\varphi(P^{(i-2)}))$ is not upper triangular mod $u^e R[\![u]\!]$. By the calculations in Lemma 2.3.12, this can happen only if one of the following two statements holds:

(1) $z_{i-1} = 1$ and $s_{i-2} \not\equiv 0 \mod v$. In this situation, s_{i-1} is a multiple of s_{i-2} mod v.
(2) $z_{i-1} = p-1$ and $r_{i-2} \not\equiv 0 \mod v$. In this situation, s_{i-1} is a multiple of r_{i-2} mod v.

Going backward, we conclude that $z_i = 0$ and $s_{i-1} \not\equiv 0$ can happen only if z_i is preceded by a subsequence $(z_{i-k-1}, z_{i-k}, ..., z_{i-1}) = (p-1, 1, ..., 1)$ with $r_{i-k-2}, s_{i-k-1}, ..., s_{i-2} \not\equiv 0 \mod v$. In other words, if $(z_i)_i$ does not contain a contiguous subsequence of the form $(p-1, 1, ..., 1, 0)$, we can always obtain F_i' via a left unipotent action on F_i.

On the other hand, if there exist $k \geq 0$ and $i \in \mathbb{Z}$ such that $(z_{i-k-1}, z_{i-k}, ..., z_{i-1}, z_i) = (p-1, 1, ..., 1, 0)$, we may choose F_j's so that $\mathcal{G}(F_j) = \mathrm{I}_\eta$ for all j. Choose F_{i-k-2} so that $b_{i-k-2} \neq 0$ and F_{i-k-1} so that c_{i-k-1} is a unit. By Lemma 2.3.12, $\mathcal{B}(M_{i-k-2}\varphi(P^{(i-k-3)}))$ must be lower triangular mod $u^e R[\![u]\!]$. Therefore, r_{i-k-2} is a unit multiple of b_{i-k-2} mod v. In turn, s_{i-k-1} is a unit times c_{i-k-1}^2 times r_{i-k-2} mod v. Inductively, we see that s_{i-1} is a unit times b_{i-k-2} mod v, and therefore, non-zero mod v. Thus, no unipotent action can give F_i' from F_i. □

Proposition 2.4.8 motivates the following definition.

Definition 2.4.10. *We say that a tame principal series $\mathbb{F}$-type τ faces the second obstruction if $(z_i)_{i\in\mathbb{Z}}$ contains a contiguous subsequence $(p-1,1,...,1,0)$ of length ≥ 2, with the number of 1's allowed to be zero.*

Our next step is to analyze when left unipotent action of the type described in Proposition 2.4.8 can be functorially associated to inertial base change data. The eventual goal is to quotient the data of Frobenius matrices by unipotent action, and encode that as a point of the stack of Breuil-Kisin modules. In particular, the unipotent action will be encoded as base change data.

For each i, let (e_i, f_i) be an inertial basis of $\mathfrak{M}_i$. The η'-eigenspace of $\mathfrak{M}_i$ is a free module over $R[\![v]\!]$ with an ordered basis given by $(u^{e-\gamma_i}e_i, f_i)$. The η'-eigenspace of $\varphi^*\mathfrak{M}_i$ is a free module over $R[\![v]\!]$ with an ordered basis given by $(u^{e-\gamma_{i+1}} \otimes e_i, 1 \otimes f_i)$. Written with respect to our choice of inertial bases, let the i-th Frobenius matrix be given as follows:

$$F_i = \begin{pmatrix} a_i & u^{e-\gamma_i}b_i \\ u^{\gamma_i}c_i & d_i \end{pmatrix}.$$

Let $\{P_i\}_i$ be a set of inertial base change matrices, where

$$P_i = \begin{pmatrix} q_i & u^{e-\gamma_i}r_i \\ u^{\gamma_i}s_i & t_i \end{pmatrix}.$$

The Frobenius map F_i, when restricted to the η'-eigenspace part and written with respect to the ordered η'-eigenspace basis of $\varphi^*\mathfrak{M}_{i-1}$ and $\mathfrak{M}_i$ has the following matrix:

$$G_i = \begin{pmatrix} a_i & b_i \\ vc_i & d_i \end{pmatrix}. \tag{2.4.2}$$

Base change of G_i is given by:

$$J_i^{-1}G_i\left(\mathrm{Ad}\begin{pmatrix} v^{p-1-z_i} & 0 \\ 0 & 1 \end{pmatrix}(\varphi(J_{i-1}))\right), \tag{2.4.3}$$

where the matrices J_i are defined as follows:

$$J_i = \begin{pmatrix} q_i & r_i \\ vs_i & t_i \end{pmatrix}.$$

Definition 2.4.11. *When a choice of an inertial basis for each i is understood, $(G_i)_i$ and $(J_i)_i$ as above will be called the Frobenius and base change matrices (respectively) for the η'-eigenspace.*

We say that the G_i's are in CDM form if the F_i's, which are the matrices for the unrestricted Frobenius maps, are in CDM form (see Definition 2.3.5).

It is clear that knowing the data of Frobenius and base change on the η'-eigenspace part is equivalent to knowing it for the entire Breuil-Kisin module.

Proposition 2.4.12. *Fix an inertial basis (e_i, f_i) for each i. Suppose that each F_i is of the form*

$$\begin{pmatrix} va_i & u^{e-\gamma_i}b_i \\ u^{\gamma_i}c_i & d_i \end{pmatrix}$$

with $a_i, b_i, c_i, d_i \in R$. For each i, denote by G_i the Frobenius matrices for restriction to η'-eigenspaces. Therefore, $G_i = \begin{pmatrix} va_i & b_i \\ vc_i & d_i \end{pmatrix}$. Let $U_i = \begin{pmatrix} 1 & y_i \\ 0 & 1 \end{pmatrix}$ for each $i \in \mathbb{Z}/f\mathbb{Z}$.

Then, whenever τ does not face the second obstruction (Definition 2.4.10), there exists a way to functorially construct inertial-base-change matrices

$$P_i = \begin{pmatrix} q_i & u^{e-\gamma_i}r_i \\ u^{\gamma_i}s_i & t_i \end{pmatrix}$$

satisfying $F_i = P_i^{-1}(U_i \star F_i)\varphi(P_{i-1})$ where $U_i \star F_i$ is as defined in Proposition 2.4.8. Equivalently,

$$G_i = J_i^{-1}U_iG_i\left(Ad\begin{pmatrix} v^{p-1-z_i} & 0 \\ 0 & 1 \end{pmatrix}(\varphi(J_{i-1}))\right)$$

where $J_i = \begin{pmatrix} q_i & r_i \\ vs_i & t_i \end{pmatrix}$.

Proof. We will build J_i as a v-adic limit of a sequence $J_i^{(n)}$. First, let $J_i^{(0)}$ be the identity matrix and define $J_{i+1}^{(n+1)}$ to be

$$J_{i+1}^{(n+1)} = U_{i+1}G_{i+1}\left(\mathrm{Ad}\begin{pmatrix} v^{p-1-z_{i+1}} & 0 \\ 0 & 1 \end{pmatrix}(\varphi(J_i^{(n)}))\right)G_{i+1}^{-1},$$

where we are inverting G_{i+1} in $\mathrm{GL}_2(R((u)))$. Therefore,

$$J_{i+1}^{(n+1)} - J_{i+1}^{(n)} = U_{i+1}G_{i+1}\left(\mathrm{Ad}\begin{pmatrix} v^{p-1-z_{i+1}} & 0 \\ 0 & 1 \end{pmatrix}(\varphi(J_i^{(n)} - J_i^{(n-1)}))\right)G_{i+1}^{-1}.$$

Let $D_i = a_id_i - b_ic_i$. Evidently, $J_{i+1}^{(1)} = U_{i+1}$. Further,

$$J_i^{(2)} - J_i^{(1)} = v^{p-z_i}\frac{y_{i-1}}{D_i}\begin{pmatrix} -a_ic_i - y_ic_i^2 & a_i^2 + y_ia_ic_i \\ -c_i^2 & a_ic_i \end{pmatrix}.$$

For $X \in M_2(R[\![v]\!])$, denote by $\mathrm{val}_v(X)$ the highest power of v that divides X. Let $\alpha_i = \mathrm{val}_v(J_i^{(2)} - J_i^{(1)})$. Then $\alpha_i \geq p - z_i$.

Now we compute the dependence of the valuation of $J_{i+1}^{(n)} - J_{i+1}^{(n-1)}$ on $J_i^{(n)} - J_i^{(n-1)}$.

If v^r divides $J_i^{(n)} - J_i^{(n-1)}$, then $v^{pr-(p-1-z_{i+1})}$ divides $\mathrm{Ad}\begin{pmatrix} v^{p-1-z_{i+1}} & 0 \\ 0 & 1 \end{pmatrix}\varphi(J_i^{(n)} - J_i^{(n-1)})$. After taking into account an extra factor of v coming from the determinant of G_{i+1} which we will need to divide by when inverting G_{i+1}, we conclude that $v^{pr-(p-z_{i+1})} = v^{p(r-1)+z_{i+1}}$ divides $J_{i+1}^{(n+1)} - J_{i+1}^{(n)}$.

Therefore,

$$\mathrm{val}_v(J_{i+1}^{(3)} - J_{i+1}^{(2)}) \quad \geq p(\alpha_i - 1) + z_{i+1},$$

$$\begin{aligned}
\mathrm{val}_v(J^{(4)}_{i+1} - J^{(3)}_{i+1}) &\geq p^2(\alpha_{i-1} - 1) + (pz_i + z_{i+1}) - p, \\
\mathrm{val}_v(J^{(5)}_{i+1} - J^{(4)}_{i+1}) &\geq p^3(\alpha_{i-2} - 1) + (p^2 z_{i-1} + pz_i + z_{i+1}) - (p^2 + p), \\
&\dots \\
\mathrm{val}_v(J^{(n)}_{i+1} - J^{(n-1)}_{i+1}) &\geq p^{n-2}(\alpha_{i-(n-3)} - 1) + \sum_{j=1}^{n-3} p^j (z_{i-(j-1)} - 1) + z_{i+1} \\
&= p^{n-2}(p - 1 - z_{i-(n-3)}) + \sum_{j=1}^{n-3} p^j (z_{i-(j-1)} - 1) + z_{i+1}.
\end{aligned}$$

We have the following scenarios:

- Suppose $z_{i-(n-3)} < p - 1$. Let $m := \lfloor \frac{n-3}{f} \rfloor$. Then

$$\begin{aligned}
\mathrm{val}_v(J^{(n)}_{i+1} - J^{(n-1)}_{i+1}) &\geq p^{n-2} + \sum_{j=1}^{n-3} p^j (z_{i-j+1} - 1) + z_{i+1} \\
&= p + \sum_{j=1}^{n-3} (p-1)p^j + \sum_{j=1}^{n-3} p^j (z_{i-(j-1)} - 1) + z_{i+1} \\
&= p + \sum_{j=1}^{n-3} (p-2)p^j + \sum_{j=0}^{n-3} p^j z_{i+1-j} \\
&\geq \sum_{j=1}^{n-3} (p-2)p^j + \sum_{k=0}^{m-1} p^k \gamma_{i+1} && \text{(using (1.4.1))} \\
&> p^{m-1}.
\end{aligned}$$

- Suppose $z_{i-(n-3)} = p - 1$ and $z_j \neq 0$ for each j. Then

$$\begin{aligned}
\mathrm{val}_v(J^{(n)}_{i+1} - J^{(n-1)}_{i+1}) &\geq \sum_{j=1}^{n-3} p^j (z_{i-(j-1)} - 1) + z_{i+1} \\
&\geq \sum_{\substack{j \in [1, n-3] \text{ and} \\ j-1 \equiv n-3 \mod f}} p^j \\
&\geq p^{n-2-f}.
\end{aligned}$$

  The second to last step uses $p > 2$.

- Suppose $z_{i-(n-3)} = p - 1$ and there exists a $k \in [0, n-3]$ such that $z_{i-(k-1)} = 0$. Take k to be as large as possible. As $\{z_j\}_j$ is f-periodic, $k \in [n-2-f, n-3]$. Since τ does not face the second obstruction, there exists a largest possible $l \in (k, n-3)$ such that $z_{i-(l-1)} > 1$. Then

$$\mathrm{val}_v(J^{(n)}_{i+1} - J^{(n-1)}_{i+1}) \geq \sum_{j=k+1}^{l} p^j (z_{i-(j-1)} - 1) + \sum_{j=1}^{k} (z_{i-(j-1)} - 1) + z_{i+1}$$

$$
\begin{aligned}
&> p^{k+1} - \sum_{j=1}^{k} p^j \\
&= p + \sum_{j=1}^{k} (p-1)p^j - \sum_{j=1}^{k} p^j \\
&= p + \sum_{j=1}^{k} (p-2)p^j \\
&> p^k \\
&\geq p^{n-2-f}.
\end{aligned}
$$

The second to last step uses that $p > 2$.

The above calculations show that whenever τ does not face the second obstruction, $(J_{i+1}^{(n)})_n$ is a Cauchy sequence for all i.

We set $J_i = \lim_{n\to\infty}(J_i^{(n)})_n$, and construct the base change matrices P_i using the data of J_i. Since $J_i^{(1)} \in M_2(R[\![v]\!])$ and $\mathrm{val}_v(J_i^{(n)} - J_i^{(n-1)}) \geq 0$ for each i and $n \geq 2$, $J_i \in M_2(R[\![v]\!])$. The same considerations apply to the inverse of J_i, showing that for each i, $J_i \in \mathrm{GL}_2(R[\![v]\!])$. Since v divides the upper and lower left entries of G_i, it can be shown by direct computation that the lower left entry of J_i is 0 mod v. Therefore, $P_i \in \mathrm{GL}_2(R[\![u]\!])$ for each i. ☐

Definition 2.4.13. *Denote the inverses of J_i's constructed in Proposition 2.4.12 by $J_i^{-1} = \mathcal{F}_i(\mathbf{U})$ to indicate the functorial dependence on the tuple of unipotent matrices $\mathbf{U} = (U_j)_j$. Then $(\mathcal{F}_i(\mathbf{U}))_i$ capture the base change data to go from $(G_i)_i \to (U_iG_i)_i$.*

3. A component of $\mathcal{C}^{\tau,\mathrm{BT}}$ as a quotient of a scheme

At this point, via Proposition 2.3.7, we have an easy way of describing the Frobenius maps for certain Breuil-Kisin modules by writing the matrices in CDM form (see Definition 2.3.5). We also have a complete description of base changes between such Frobenius matrices in Proposition 2.4.1. Finally, in some cases, we have a way of obtaining Frobenius matrices in CDM form through a particular group action (see Proposition 2.4.8). The goal of this section is to use these results to write a certain irreducible component of $\mathcal{C}^{\tau,\mathrm{BT}}$ (Definition 2.2.3) as a quotient stack $[X/G]$ for some scheme X and group scheme G acting on X. We will use this presentation to compute global functions on the component.

In order to allow us to use Propositions 2.3.7, 2.4.1 and 2.4.8, we make the following assumption for the entirety of this section.

Assumption 3.0.1. *The tame principal series $\mathbb{F}$-type $\tau = \eta \oplus \eta'$ satisfies:*

- *$\eta \neq \eta'$, and*
- *τ does not face either the first obstruction (in the sense of Definition 2.4.7) or the second obstruction (in the sense of Definition 2.4.10).*

3.1. **A smooth map from a scheme to $\mathcal{C}$.** Let $G = (\mathbb{G}_m)_{\mathbb{F}}^{f+1} \times_{\mathbb{F}} U_{\mathbb{F}}^f$ and $X = (\mathrm{GL}_2)_{\mathbb{F}} \times_{\mathbb{F}} (\mathrm{SL}_2)_{\mathbb{F}}^{f-1}$, where $U \cong \mathbb{G}_a$ is the upper unipotent subgroup of GL_2. Define a G-action on X in the following way:

Let $(\lambda, \mu, r_1, r_2, ..., r_{f-1}, m_0, ..., m_{f-1}) \in G$ and $(A_0, ..., A_{f-1}) \in X$. Then

$$(\lambda, \mu, r_1, r_2, ..., r_{f-1}, m_0, ..., m_{f-1}) \cdot (A_0, ..., A_{f-1}) := \tag{3.1.1}$$

$$\Bigg(\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \mu^{-1} \end{pmatrix} m_0 A_0 \begin{pmatrix} r_{f-1}^{-1} & 0 \\ 0 & r_{f-1} \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix},$$

$$\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \mu^{-1} \end{pmatrix} \begin{pmatrix} r_1 & 0 \\ 0 & r_1^{-1} \end{pmatrix} m_1 A_1 \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix},$$

$$\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \mu^{-1} \end{pmatrix} \begin{pmatrix} r_2 & 0 \\ 0 & r_2^{-1} \end{pmatrix} m_2 A_2 \begin{pmatrix} r_1^{-1} & 0 \\ 0 & r_1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix},$$

$$\cdots$$

$$\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \mu^{-1} \end{pmatrix} \begin{pmatrix} r_{f-1} & 0 \\ 0 & r_{f-1}^{-1} \end{pmatrix} m_{f-1} A_{f-1} \begin{pmatrix} r_{f-2}^{-1} & 0 \\ 0 & r_{f-2} \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \Bigg).$$

Definition 3.1.1. *Define a functor $\mathcal{T} : X \to \mathcal{C}^\tau$ by sending*

$$\left(\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}, \ldots, \begin{pmatrix} a_{f-1} & b_{f-1} \\ c_{f-1} & d_{f-1} \end{pmatrix} \right) \in X(R)$$

to the Breuil-Kisin module $\mathfrak{M} \in \mathcal{C}^\tau(R)$ constructed as follows:

(1) $\mathfrak{M}_i = R[\![u]\!]e_i \oplus R[\![u]\!]f_i$.
(2) *With respect to the basis $\{e_i, f_i\}$, the action of $g \in Gal(K'/K)$ is given by the diagonal matrix $\begin{pmatrix} \eta(g) & 0 \\ 0 & \eta'(g) \end{pmatrix}$.*
(3) *With respect to the basis $\{u^{e-\gamma_i} \otimes e_{i-1}, 1 \otimes f_{i-1}\}$ (resp. $\{u^{e-\gamma_i}e_i, f_i\}$) of the η'-eigenspace of $\varphi^*\mathfrak{M}_{i-1}$ (resp. $\mathfrak{M}_i$), the matrix of the restriction of the i-th Frobenius map $\varphi^*\mathfrak{M}_{i-1} \to \mathfrak{M}_i$ to the η'-eigenspace is $\begin{pmatrix} va_i & b_i \\ vc_i & d_i \end{pmatrix}$.*

Consider the pullback of $\mathcal{T}$ by the closed embedding $\mathcal{C}^{\tau,\mathrm{BT}} \hookrightarrow \mathcal{C}^\tau$. The pullback is a closed subscheme of X that contains all the closed points of X by Lemma 2.3.2. Since X is reduced, the pullback must be all of X and $\mathcal{T}$ must map X into $\mathcal{C}^{\tau,\mathrm{BT}}$. Choose an irreducible component $\mathcal{X}(\tau) \subset \mathcal{C}^{\tau,\mathrm{BT}}$ containing the image of $\mathcal{T}$. Such an irreducible component must exist because X is irreducible, although a priori, it is not unique (we will see later in Proposition 3.1.4 that in fact it is unique). Henceforth, we will see $\mathcal{T}$ as a functor from X to $\mathcal{X}(\tau)$.

Definition 3.1.2. *Suppose τ satisfies Assumption 3.0.1. We define a functor $F : G \times X \to X \times_{\mathcal{X}(\tau)} X$ in the following way:*

Let $g = (\lambda, \mu, r_1, ..., r_{f-1}, m_0, ..., m_{f-1}) \in G(R)$ and $x \in X(R)$. Then $F((g, x))$ is the triple $(x,\ g \cdot x,\ \{J_i\}_i)$ where $(x,\ g \cdot x) \in X(R) \times X(R)$ and $\{J_i\}_i$ are base change matrices for η'-eigenspaces (in the sense of Definition 2.4.11) that encode transformation of the Frobenius matrices of $\mathcal{T}(x)$ to those of $\mathcal{T}(g \cdot x)$. They are given by:

$$J_i := \begin{cases} \mathcal{F}_i((m_j)_j) \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} & \text{for } i = 0, \\ \mathcal{F}_i((m_j)_j) \begin{pmatrix} r_i^{-1} & 0 \\ 0 & r_i \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} & \text{for } i \in \{1, \dots, f-1\}. \end{cases} \tag{3.1.2}$$

Here, $\mathcal{F}_i((m_j)_j)$ are described in Definition 2.4.13.

Definition 3.1.3. *Suppose τ satisfies Assumption 3.0.1. We let $\widetilde{\mathcal{T}} : [X/G] \to \mathcal{X}(\tau)$ be the functor induced by F.*

Proposition 3.1.4. *The functor $\widetilde{\mathcal{T}}$ is an isomorphism.*

The proof of Proposition 3.1.4 will be given in several steps outlined below.

Lemma 3.1.5. *The functor F in Definition 3.1.2 is surjective on points valued in Artinian local $\mathbb{F}$-algebras and a monomorphism.*

Proof. Let R be an Artinian local $\mathbb{F}$-algebra. Let $(x, y, \{J_i\}_i) \in (X \times_{\mathcal{C}^\tau} X)(R)$ where $(x, y) \in X(R) \times X(R)$ and $\{J_i\}_i$ are the base change matrices for η'-eigenspaces to transform $\mathcal{T}(x)$ to $\mathcal{T}(y)$.

Let $(A_i)_{i=0}^{f-1}$ be the Frobenius matrices for the η'-eigenspace for $\mathcal{T}(x)$. Because τ does not face the first obstruction, $\mathcal{T}(x)$ is not of bad genre and with respect to a suitable choice of inertial bases, the Frobenius matrices of $\mathcal{T}(x)$ will be in CDM form (see Proposition 2.3.7 and Definition 2.4.11). Because τ also does not face the second obstruction, using Proposition 2.4.8 we can uniquely determine $(r_1, \dots, r_{f-1}) \in \mathbb{G}_m^{f-1}(R)$ and $(m_0, \dots, m_{f-1}) \in U^f(R)$ so that the tuple $(A_i')_{i=0}^{f-1}$ defined below is in CDM form:

$$A_i' := \begin{cases} m_i A_i \begin{pmatrix} r_{i-1}^{-1} & 0 \\ 0 & r_{i-1} \end{pmatrix} & \text{if } i = 0, \\ \begin{pmatrix} r_i & 0 \\ 0 & r_i^{-1} \end{pmatrix} m_i A_i & \text{if } i = 1, \\ \begin{pmatrix} r_i & 0 \\ 0 & r_i^{-1} \end{pmatrix} m_i A_i \begin{pmatrix} r_{i-1}^{-1} & 0 \\ 0 & r_{i-1} \end{pmatrix} & \text{if } i \in \{2, \dots, f-1\}. \end{cases}$$

Similarly, let $(B_i)_{i=0}^{f-1}$ be the Frobenius matrices for the η'-eigenspace corresponding to the data of $\mathcal{T}(y)$. We can uniquely determine $(s_1, \dots, s_{f-1}) \in \mathbb{G}_m^{f-1}(R)$ and $(n_0, \dots, n_{f-1}) \in U^{f-1}(R)$ so that the tuple $(B_i')_{i=0}^{f-1}$ defined below is in CDM form:

$$B_i' = \begin{cases} n_i B_i \begin{pmatrix} s_{i-1}^{-1} & 0 \\ 0 & s_{i-1} \end{pmatrix} & \text{if } i = 0, \\ \begin{pmatrix} s_i & 0 \\ 0 & s_i^{-1} \end{pmatrix} n_i B_i & \text{if } i = 1, \\ \begin{pmatrix} s_i & 0 \\ 0 & s_i^{-1} \end{pmatrix} n_i B_i \begin{pmatrix} s_{i-1}^{-1} & 0 \\ 0 & s_{i-1} \end{pmatrix} & \text{if } i \in \{2, \dots, f-1\}. \end{cases}$$

Since $(A_i')_i$ and $(B_i')_i$ are base changes of $(A_i)_i$ and $(B_i)_i$ respectively, there exist base change matrices $(P_i)_i$ that allow us to transform $(A_i')_i$ to $(B_i')_i$. By Proposition 2.4.1, there exist $\lambda, \mu \in \mathbb{G}_m(R)$ so that $P_0 = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)$ and for $i \in \{1, \dots, f-1\}$, $P_i = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)\left(\begin{smallmatrix} k_i & 0 \\ 0 & k_i^{-1} \end{smallmatrix}\right)$, where $k_i = \mu\lambda^{-1}$ if $|\{j \in [1, i] \mid \mathcal{G}(A_j') = \mathrm{II}\}|$ is odd, and 1 otherwise.

We now use $(r_i)_i$, $(s_i)_i$, $(m_i)_i$, $(n_i)_i$ and $(P_i)_i$ to write $(B_i)_i$ in terms of $(A_i)_i$.

$$B_i = \begin{cases} n_i^{-1} \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \mu^{-1} \end{pmatrix} m_i A_i \begin{pmatrix} r_{i-1}^{-1} & 0 \\ 0 & r_{i-1} \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \\ \qquad \begin{pmatrix} k_{i-1} & 0 \\ 0 & k_{i-1}^{-1} \end{pmatrix} \begin{pmatrix} s_{i-1} & 0 \\ 0 & s_{i-1}^{-1} \end{pmatrix} & \text{for } i = 0, \\ n_i^{-1} \begin{pmatrix} s_i^{-1} & 0 \\ 0 & s_i \end{pmatrix} \begin{pmatrix} k_i^{-1} & 0 \\ 0 & k_i \end{pmatrix} \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \mu^{-1} \end{pmatrix} \\ \qquad \begin{pmatrix} r_i & 0 \\ 0 & r_i^{-1} \end{pmatrix} m_i A_i \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} & \text{for } i = 1, \\ n_i^{-1} \begin{pmatrix} s_i^{-1} & 0 \\ 0 & s_i \end{pmatrix} \begin{pmatrix} k_i^{-1} & 0 \\ 0 & k_i \end{pmatrix} \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \mu^{-1} \end{pmatrix} \begin{pmatrix} r_i & 0 \\ 0 & r_i^{-1} \end{pmatrix} \\ \qquad m_i A_i \begin{pmatrix} r_{i-1}^{-1} & 0 \\ 0 & r_{i-1} \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} k_{i-1} & 0 \\ 0 & k_{i-1}^{-1} \end{pmatrix} \begin{pmatrix} s_{i-1} & 0 \\ 0 & s_{i-1}^{-1} \end{pmatrix} & \text{for } i \in \{2, \dots, f-1\}. \end{cases}$$

Simplifying,

$$B_i = \begin{cases} \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \mu^{-1} \end{pmatrix} \tilde{m}_i A_i \begin{pmatrix} s_{i-1}k_{i-1}r_{i-1}^{-1} & 0 \\ 0 & s_{i-1}^{-1}k_{i-1}^{-1}r_{i-1} \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} & \text{for } i = 0, \\ \\ \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \mu^{-1} \end{pmatrix} \begin{pmatrix} s_i^{-1}k_i^{-1}r_i & 0 \\ 0 & s_ik_ir_i^{-1} \end{pmatrix} \tilde{m}_i A_i \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} & \text{for } i = 1, \\ \\ \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \mu^{-1} \end{pmatrix} \begin{pmatrix} s_i^{-1}k_i^{-1}r_i & 0 \\ 0 & s_ik_ir_i^{-1} \end{pmatrix} \tilde{m}_i A_i & \\ \qquad \begin{pmatrix} s_{i-1}k_{i-1}r_{i-1}^{-1} & 0 \\ 0 & s_{i-1}^{-1}k_{i-1}^{-1}r_{i-1} \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} & \text{for } i \in \{2, \ldots, f-1\}. \end{cases}$$

where $\tilde{m}_i$ are suitably chosen unipotent matrices.

This implies the existence of a $g \in G$ such that $y = g \cdot x$. By (3.1.2), $F((g, x))$ contains the data of some base change matrices to go from $\{A_i\}$ to $\{B_i\}$. These can only differ by a fixed scalar multiple from the original base change matrices $\{J_i\}_i$ (by Corollary 2.4.5). Scaling λ and μ by this fixed multiple gives us a g' such that $F((g', x)) = (x, y, \{J_i\}_i)$. This shows surjectivity on Artinian local points.

Now suppose that R is *any* $\mathbb{F}$-algebra. Let $(g, x), (g', x') \in (G \times X)(R)$ such that $F((g, x)) = F((g', x')) = (x, y, \{J_i\}_i)$. Then $x = x'$ and $y = g \cdot x = g' \cdot x$. Let $(A_i)_{i=0}^{f-1}$ be the Frobenius matrices for η'-eigenspaces in the data of $\mathcal{T}(x)$ (described in Definition 3.1.1) and $(B_i)_{i=0}^{f-1}$ be the corresponding matrices for $\mathcal{T}(y)$. Let

$$g = (\lambda, \mu, r_1, r_2, ..., r_{f-1}, m_0, ..., m_{f-1}),$$
$$g' = (\lambda', \mu', r'_1, r'_2, ..., r'_{f-1}, m'_0, ..., m'_{f-1}).$$

By (3.1.2),

$$J_0 = \mathcal{F}_0((m_j)_j) \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} = \mathcal{F}_0((m'_j)_j) \begin{pmatrix} \lambda' & 0 \\ 0 & \mu' \end{pmatrix}.$$

All inertial base change matrices for η'-eigenspaces, including $\mathcal{F}_i((m_j)_j)$, are upper unipotent mod v. Reducing mod v, we get $\lambda = \lambda'$ and $\mu = \mu'$.

For $i \in \{1, \ldots, f-1\}$,

$$J_i = \mathcal{F}_i((m_j)_j) \begin{pmatrix} r_i^{-1} & 0 \\ 0 & r_i \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} = \mathcal{F}_i((m'_j)_j) \begin{pmatrix} r_i'^{-1} & 0 \\ 0 & r'_i \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}.$$

Again reducing mod v, we get $(r_1, ..., r_{f-1}) = (r'_1, ..., r'_{f-1})$. Finally we use (3.1.1) to write B_i in terms of A_i and g, and compare it to B_i written in terms of A_i and g'. It is immediate that for each i, $m_i = m'_i$. □

Lemma 3.1.6. *The functor F is an isomorphism.*

Proof. We note that the diagonal of F is an isomorphism because F is a monomorphism (by Lemma 3.1.5). This implies via [Sta18, Tag 0AHJ] that F is representable by algebraic spaces.

To show F is an isomorphism, we will show that F is étale since it is already known to be a surjective monomorphism and étale monomorphisms are open immersions. The property of being étale is étale-smooth local on the source-and-target by [Sta18, Tag 0CG3]. Therefore, it suffices by [Sta18, Tag 0CIF] to show the top arrow in the following diagram is étale, where T is a smooth cover of $X \times_{\mathcal{X}(\tau)} X$ and $f : W \to T \times_{(X \times_{\mathcal{X}(\tau)} X)} (G \times_{\mathbb{F}} X)$ is an étale cover.

$$\begin{tikzcd}
W \arrow[r, "f"] & T \times_{(X \times_{\mathcal{X}(\tau)} X)} (G \times_{\mathbb{F}} X) \arrow[r] \arrow[d] & T \arrow[d] \\
 & G \times_{\mathbb{F}} X \arrow[r, "F"] & X \times_{\mathcal{X}(\tau)} X
\end{tikzcd}$$

The functor F is unramified because it is locally of finite presentation with its diagonal an isomorphism. The only thing remaining to check then is that the map $W \xrightarrow{f} T \times_{(X \times_{\mathcal{X}(\tau)} X)} (G \times_{\mathbb{F}} X) \xrightarrow{pr_1} T$ is formally smooth. By [Sta18, Tag 02HX], we may test formal smoothness of this map using Artinian local rings. As $W \xrightarrow{f} T \times_{(X \times_{\mathcal{X}(\tau)} X)} (G \times_{\mathbb{F}} X)$ is already known to be formally smooth, we need only check the lifting property for $T \times_{(X \times_{\mathcal{X}(\tau)} X)} (G \times_{\mathbb{F}} X) \xrightarrow{pr_1} T$. This is equivalent to checking the following:

Suppose R and S are Artinian local $\mathbb{F}$-algebras with $j : Spec\ S \to Spec\ R$ a closed scheme and $j^\# : R \to S$ a surjection of local rings with the kernel squaring to zero. Then the dashed arrow exists in the following diagram

$$\begin{tikzcd}
Spec\ S \arrow[r, "j"] \arrow[d] & Spec\ R \arrow[d] \arrow[dl, dashed] \\
G \times_{\mathbb{F}} X \arrow[r, "F"] & X \times_{\mathcal{X}(\tau)} X
\end{tikzcd} .$$

Since F induces a bijection for points valued in Artinian local rings (by Lemma 3.1.5), the existence of a unique dashed arrow is guaranteed. □

Let $l/\mathbb{F}$ be a field with x an l-point of X, such that $\mathfrak{M} = \mathcal{T}(x)$ is a Breuil-Kisin module over l. Then there exists a map $G \times_{\mathbb{F}} l \to l \times_{\mathcal{X}(\tau)} X \xrightarrow{\sim} \mathfrak{M} \times_{\mathcal{X}(\tau)} X$. By Lemma 3.1.5, this map is surjective on field-valued points and the fiber of $G \times_{\mathbb{F}} l \to l \times_{\mathcal{X}(\tau)} X$ over any field-valued point contains exactly one point, and is therefore of dimension 0. By [Sta18, Tag

0DS6], dimension of $l \times_{\mathcal{X}(\tau)} X \xrightarrow{\sim} \mathfrak{M} \times_{\mathcal{X}(\tau)} X = \dim(G \times_{\mathbb{F}} l) = \dim G$. Since the fiber over $\mathfrak{M}$ in X is of the same dimension as G, the fiber over $\mathfrak{M}$ in $[X/G]$ has dimension 0.

Applying [Sta18, Tag 0DS6] again to the map $\widetilde{\mathcal{T}}$ and using the above calculations of fiber dimension over $\mathfrak{M} \in \mathcal{X}(\tau)(l)$, we obtain that the dimension of the scheme-theoretic image of $\widetilde{\mathcal{T}}$ is the same as the dimension of $[X/G]$ which is f.

Lemma 3.1.7. *Suppose τ satisfies Assumption 3.0.1.*

(1) *Let R be an arbitrary $\mathbb{F}$-algebra and $\mathfrak{M} \in \mathcal{X}(\tau)(R)$. Fix an inertial basis for each $\mathfrak{M}_i$. Let F_i denote the matrix for the Frobenius map $\varphi^*(\mathfrak{M}_{i-1}) \to \mathfrak{M}_i$ with respect to the chosen bases. Then, for each i, the top left entry of F_i is 0 mod v.*
(2) *The map $\mathcal{T}$ is a surjection onto $\mathcal{X}(\tau)$.*

Proof. Consider the substack $\mathcal{L}$ of $\mathcal{X}(\tau)$ defined in the following way: If R is any $\mathbb{F}$-algebra, then $\mathcal{L}(R) \subset \mathcal{X}(\tau)(R)$ is the subgroupoid of those Breuil-Kisin modules for which the upper left entry of the Frobenius matrices is 0 mod v when the Frobenius matrices are written with respect to any inertial bases. A direct computation shows that this property is invariant under inertial base change. We claim, first of all, that $\mathcal{L}$ is a closed substack of $\mathcal{X}(\tau)$.

We can check it is representable by algebraic spaces and a closed immersion after pulling back to an affine scheme and working *fpqc*-locally (by [Sta18, Tag 0420]). Let R be an $\mathbb{F}$-algebra and $\mathfrak{M}$ an R point of $\mathcal{X}(\tau)$. For $i \in \mathbb{Z}/f\mathbb{Z}$, choose an inertial basis $\{e_i, f_i\}$ of $\mathfrak{M}_i$, and write Frobenius matrices F_i of $\mathfrak{M}$ with respect to these bases. Suppose that for each i, the upper left entry of F_i equals a_i mod v, where $a_i \in R$. For every R-algebra S, the Frobenius matrices of $\mathfrak{M}_S$ with respect to these bases are given by $\{F_i \otimes S\}_i$. Then $\mathfrak{M}_S$ is a point of $\mathcal{L}$ if and only if $a_i = 0$ in S for each i. Therefore the pullback of $\mathcal{L} \to \mathcal{X}(\tau)$ by the map $\mathfrak{M}$: $\operatorname{Spec} R \to \mathcal{X}(\tau)$ is given by the closed immersion $V(a_0, ..., a_{f-1}) \hookrightarrow \operatorname{Spec} R$.

Secondly, we note that $\mathcal{T}$ factors as $\mathcal{T} : X \twoheadrightarrow \mathcal{L} \hookrightarrow \mathcal{X}(\tau)$, and that $\mathcal{X}(\tau)$ is reduced by construction in [CEGS19, Cor. 4.8.1]. Therefore, by dimension considerations, $\mathcal{X}(\tau)$ is the scheme-theoretic image of $\mathcal{T}$. However, the scheme-theoretic image of $\mathcal{T}$ must be contained in $\mathcal{L}$, the latter being a closed substack. Therefore, $\mathcal{L} = \mathcal{X}(\tau)$. Both assertions of the lemma follow immediately. □

Remark 3.1.8. *In the proof of surjectivity above, the fact that the dimension of $\mathcal{X}(\tau)$ is f depends on the fact that K is an unramified extension of $\mathbb{Q}_p$ (see* [CEGS19, Prop. 3.10.20]*).*

Lemma 3.1.9. *Suppose τ satisfies Assumption 3.0.1. The map $\widetilde{\mathcal{T}} : [X/G] \to \mathcal{X}(\tau)$ is an étale monomorphism, representable by an algebraic space.*

Proof. To see that $\tilde{\mathcal{T}}$ is a monomorphism and representable by an algebraic space, we show that the diagonal is an isomorphism. This is implied by the fact that the top arrow in the following cartesian diagram is an isomorphism (by Lemma 3.1.6) and [Sta18, Tag 04XD].

$$\begin{tikzcd}
G \times_{\mathbb{F}} X \arrow[r, "\mathrm{pr}_2\text{,action}"] \arrow[d] & X \times_{\mathcal{X}(\tau)} X \arrow[d] \\
{[X/G]} \arrow[r, "\Delta"] & {[X/G] \times_{\mathcal{X}(\tau)} [X/G]}
\end{tikzcd}$$

Since the diagonal is an isomorphism, we also have that $\tilde{\mathcal{T}}$ is unramified. Therefore, to show étaleness, it suffices to show that $\tilde{\mathcal{T}}$ is formally smooth [Sta18, Tag 0DP0]. As we did in Lemma 3.1.6, we may check formal smoothness using Artinian local rings, and in fact, show formal smoothness of $\mathcal{T}$ since any affine point of $[X/G]$ can be written as a composition of an affine point of X with the quotient map $X \to [X/G]$. It suffices then to show the following:

Suppose R and S are Artinian local $\mathbb{F}$-algebras with $j : Spec\, S \to Spec\, R$ a closed scheme and $j^{\#} : R \to S$ a surjection of local rings with the kernel I squaring to zero. Then the dashed arrow in the following diagram exists so that all triangles commute:

$$\begin{tikzcd}
Spec\ S \arrow[r, "j"] \arrow[d, "a"'] & Spec\ R \arrow[d, "b"] \arrow[dl, dashed] \\
X \arrow[r, "\mathcal{T}"] & \mathcal{X}(\tau)
\end{tikzcd} .$$

In order to construct such an arrow, we first claim that there exists some $c \in X(R)$ such that $\mathcal{T}(c) = b$. To see this, note that the determinant of each of the Frobenius matrices of b is divisible by v (by Lemma 3.1.7(1)). Further, modulo the maximal ideal of R, the u-adic valuation of the determinant of each Frobenius map is e (by Lemma 2.3.2). Therefore, the same holds true over R, and consequently b is of Hodge type $\mathbf{v}_0$ (see Definition 2.3.1). Moreover, again by Lemma 3.1.7(1), each Frobenius matrix is in η-form (see Definition 2.3.3). By Proposition 2.3.7, we can find a CDM form for b giving us a suitable point $c \in \mathcal{X}(\tau)(R)$.

Since $\mathcal{T}(a) = b \circ j = \mathcal{T}(c \circ j)$, there exists some $g \in G(S)$, such that $g \cdot (c \circ j) = a$ (by Lemma 3.1.5). Lift g to any $\tilde{g} \in G(R)$. Then $\tilde{g} \cdot c$ is the appropriate choice for the dashed arrow in the diagram above. □

Proof of Proposition 3.1.4. Follows from Lemmas 3.1.7 and 3.1.9. □

Proposition 3.1.10. *Suppose τ satisfies Assumption 3.0.1. The ring of global functions on $\mathcal{X}(\tau)$ is isomorphic to $\mathbb{F}[x,y][\frac{1}{y}]$.*

Proof. By Proposition 3.1.4 the global functions of $\mathcal{X}(\tau)$ are the G-invariant global functions of X, where $G = (\mathbb{G}_m)_{\mathbb{F}}^{f+1} \times_{\mathbb{F}} U_{\mathbb{F}}^f$ and $X = (\mathrm{GL}_2)_{\mathbb{F}} \times_{\mathbb{F}} (\mathrm{SL}_2)_{\mathbb{F}}^{f-1}$ and the G-action on X is as in (3.1.1). These functions are the same as the $(\mathbb{G}_m)_{\mathbb{F}}^{f+1}$-invariant global functions of $(U\backslash \mathrm{GL}_2)_{\mathbb{F}} \times_{\mathbb{F}} (U\backslash \mathrm{SL}_2)_{\mathbb{F}}^{f-1}$. By the isomorphisms

$$\begin{aligned} U\backslash \mathrm{GL}_2 &\xrightarrow{\sim} \mathbb{A}^2 \smallsetminus \{0\} \times \mathbb{G}_m, \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto ((c,d),\, ad-bc) \end{aligned}$$

and

$$\begin{aligned} U\backslash \mathrm{SL}_2 &\xrightarrow{\sim} \mathbb{A}^2 \smallsetminus \{0\}, \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto (c,d), \end{aligned}$$

the ring of global functions of $(U\backslash \mathrm{GL}_2)_{\mathbb{F}} \times_{\mathbb{F}} (U\backslash \mathrm{SL}_2)_{\mathbb{F}}^{f-1}$ is isomorphic to

$$\mathbb{F}[c_0, ..., c_{f-1}, d_0, ..., d_{f-1}, D][\frac{1}{D}]$$

where $\{c_i, d_i\}$ capture the lower two entries of the i-th matrix group while D captures the determinant of the GL_2 matrices. Under this identification, $(\lambda, \mu, r_1, r_2, ..., r_{f-1}) \in (\mathbb{G}_m)_{\mathbb{F}}^{f+1}$ acts on the global functions of $(U\backslash \mathrm{GL}_2)_{\mathbb{F}} \times_{\mathbb{F}} (U\backslash \mathrm{SL}_2)_{\mathbb{F}}^{f-1}$ via:

$$\begin{aligned} g \cdot c_i &= \begin{cases} \lambda\mu^{-1} r_{i-1}^{-1} c_i & \text{if } i = 0, \\ \lambda\mu^{-1} r_i^{-1} c_i & \text{if } i = 1, \\ \lambda\mu^{-1} r_i^{-1} r_{i-1}^{-1} c_i & \text{if } i \in \{2, \ldots, f-1\}, \end{cases} \\ g \cdot d_i &= \begin{cases} r_{i-1} d_i & \text{if } i = 0, \\ r_i^{-1} d_i & \text{if } i = 1, \\ r_i^{-1} r_{i-1} d_i & \text{if } i \in \{2, \ldots, f-1\}, \end{cases} \\ g \cdot D &= D. \end{aligned}$$

Therefore, the subring of $(\mathbb{G}_m)_{\mathbb{F}}^{f-1}$-invariant functions is $\mathbb{F}[d_0 \cdots d_{f-1}, D][\frac{1}{D}] \cong \mathbb{F}[x,y][\frac{1}{y}]$. □

3.2. **Identifying the component.** Our next order of business is to identify precisely which irreducible component of $\mathcal{C}^{\tau,\mathrm{BT}}$ can be written as the quotient stack $[X/G]$ using the strategy employed in Section 3.1. [CEGS19, Cor. 4.8.1] shows that the irreducible components of $\mathcal{C}^{\tau,\mathrm{BT}}$ are in one-to-one correspondence with subsets of $\mathbb{Z}/f\mathbb{Z}$ called shapes. We now recall the definition of the shape of a Breuil-Kisin module and some of the specifics of the correspondence between irreducible components and shapes as it applies to our situation.

Definition 3.2.1. *Let $\mathfrak{B} \in \mathcal{C}^{\tau,\mathrm{BT}}(\overline{\mathbb{F}})$ be an extension of $\mathfrak{M}$ by $\mathfrak{N}$, where $\mathfrak{M}$ and $\mathfrak{N}$ are two rank 1 Breuil-Kisin modules. For each i, let m_i be a generator of $\mathfrak{M}_i$ as an $\overline{\mathbb{F}}[\![u]\!]$ module.*

The shape of $\mathfrak{B}$ is the set $J := \{i \in \mathbb{Z}/f\mathbb{Z} \mid \forall g \in Gal(K'/K), gm_i \equiv \eta(g)m_i \mod u\}$.

Suppose the image of m_{i-1} under Frobenius is $a_i u^{r_i} m_i$, where $a_i \in \overline{\mathbb{F}}[\![u]\!]^*$. Then the refined shape of $\mathfrak{B}$ is the pair (J, r) where J is the shape of $\mathfrak{B}$ and $r = (r_i)_{i \in \mathbb{Z}/f\mathbb{Z}}$.

Definition 3.2.2. *Let $r = (r_i)_{i \in \mathbb{Z}/f\mathbb{Z}}$ be as follows:*

$$r_i = \begin{cases} e & \text{if } i-1, i \in J \text{ or if } i-1, i \notin J, \\ \gamma_i & \text{if } i-1 \in J \text{ and } i \notin J, \\ e-\gamma_i & \text{if } i-1 \notin J \text{ and } i \in J. \end{cases} \tag{3.2.1}$$

Then the refined shape (J, r) is the maximal refined shape associated to J.

By [CEGS19, Lem. 4.5.13], the irreducible component $\mathcal{C}^{\tau,\mathrm{BT}}(J)$ is the closure of a constructible set whose $\overline{\mathbb{F}}$ points are precisely the Breuil-Kisin modules of maximal refined shape associated to J.

Lemma 3.2.3. *Let $\tau = \eta \oplus \eta'$ be a tame principal series $\mathbb{F}$-type with $\eta \neq \eta'$. Let $J \subset \mathbb{Z}/f\mathbb{Z}$. Then $\mathcal{C}^{\tau,\mathrm{BT}}(J)(\overline{\mathbb{F}})$ contains a dense open subset of Breuil-Kisin modules $\mathfrak{B}$ satisfying $\mathcal{G}(\mathfrak{B}_i) = \mathrm{I}_\eta$ for each i if and only if $J = \mathbb{Z}/f\mathbb{Z}$.*

Proof. Let $\mathfrak{B}$ be an $\overline{\mathbb{F}}$ point of maximal refined shape associated to J. Let $\mathfrak{B}$ be an extension of $\mathfrak{M}$ by $\mathfrak{N}$ where $\mathfrak{M}$ and $\mathfrak{N}$ are two rank 1 Breuil-Kisin modules. For each i, choose a generator m_i of $\mathfrak{M}_i$ and n_i of $\mathfrak{N}_i$ as $\overline{\mathbb{F}}[\![u]\!]$-modules. Let the image under Frobenius of m_{i-1} be $a_i u^{r_i} m_i$ and that of n_{i-1} be $a'_i u^{r'_i} n_i$ for some $a_i, a'_i \in \overline{\mathbb{F}}[\![u]\!]^*$. The strong determinant condition forces that $r_i + s'_i = e$ for each i by Lemma 2.3.2. By making careful choices of m_i and n_i (using either [CEGS19, Lem. 4.4.1] or the proof of Lemma 2.1.3), we can construct an inertial basis of $\mathfrak{B}_i$ made up of m_i and a lift $\tilde{n}_i$ of n_i. Now we use the explicit description of r_i in (3.2.1) to check the genre of the Frobenius maps for $\mathfrak{B}$ for different J's. We have the following possibilities:

(1) If $J = \mathbb{Z}/f\mathbb{Z}$, then $Gal(K'/K)$ acts on m_i via η (and therefore on n_i via η') for each $i \in \mathbb{Z}/f\mathbb{Z}$. Since $r_i = e$, the genre of $\mathfrak{B}_i$ is I_η for each i.
(2) If J is the empty set, $Gal(K'/K)$ acts on n_i via η for each $i \in \mathbb{Z}/f\mathbb{Z}$. As $r'_i = 0$, the genre of $\mathfrak{B}_i$ is $\mathrm{I}_{\eta'}$ for each i.
(3) If J is neither $\mathbb{Z}/f\mathbb{Z}$ nor empty, then there exists an $i \in \mathbb{Z}/f\mathbb{Z}$ such that $i \in J$, but $i+1 \notin J$. This implies that $r_{i+1} = \gamma_{i+1}$ and $r'_{i+1} = e - \gamma_{i+1}$. Consider the Frobenius matrix for $\varphi^*\mathfrak{B}_i \to \mathfrak{B}_{i+1}$ with respect to inertial bases $(m_i, \tilde{n}_i)$ of $\mathfrak{B}_i$ and $(\tilde{n}_{i+1}, m_{i+1})$ of $\mathfrak{B}_{i+1}$. The matrix has a zero in the lower right corner, and therefore is of genre $\mathrm{I}_{\eta'}$ or of genre II. Either way, $\mathcal{G}(\mathfrak{B}_{i+1}) \neq \mathrm{I}_\eta$.

∎

Corollary 3.2.4. *Recall $\widetilde{\mathcal{T}}$ from Definition 3.1.3. The scheme-theoretic image of $\widetilde{\mathcal{T}}$ is $\mathcal{C}^{\tau,\mathrm{BT}}(\mathbb{Z}/f\mathbb{Z})$.*

Proof. Since $\widetilde{\mathcal{T}} : [X/G] \to \mathcal{X}(\tau)$ is an isomorphism (by Proposition 3.1.4), there exists a dense open set of $\mathcal{X}(\tau)$ having the following property: If $\mathfrak{B}$ is an $\overline{\mathbb{F}}$ point of this dense open, then the lower right entry of each of its Frobenius matrices (with respect to inertial bases) is invertible. In other words, each Frobenius map has genre I_η. By Lemma 3.2.3, $\mathcal{X}(\tau)$ must be $\mathcal{C}^{\tau,\mathrm{BT}}(\mathbb{Z}/f\mathbb{Z})$. □

Corollary 3.2.5. *Let τ be a tame principal series $\mathbb{F}$-type satisfying Assumption 3.0.1. Then the ring of global functions on $\mathcal{C}^{\tau,\mathrm{BT}}(\mathbb{Z}/f\mathbb{Z})$ is isomorphic to $\mathbb{F}[x,y][\frac{1}{y}]$.*

Proof. It follows from Proposition 3.1.10 and Corollary 3.2.4. □

4. Passage to the Emerton-Gee stack

4.1. **Image of irreducible components of $\mathcal{C}^{\tau,\mathrm{BT}}$ in $\mathcal{Z}$.** Given a tame principal series $\mathbb{F}$-type τ, $\mathcal{Z}^\tau$ is the scheme-theoretic image of $\mathcal{C}^{\tau,\mathrm{BT}}$ in $\mathcal{Z}$ (Definition 2.2.4). By [CEGS19, Prop. 3.10.19], $\mathcal{Z}^\tau$ is of pure dimension $[K : \mathbb{Q}_p]$. [CEGS19, Cor. 4.8.3] tells us that the irreducible components of $\mathcal{Z}^\tau$ are indexed by shapes $J \in \mathcal{P}_\tau$, where $\mathcal{P}_\tau$ is defined in the following way.

Definition 4.1.1. *For a tame principal series $\mathbb{F}$-type $\tau = \eta \oplus \eta'$, let $\mathcal{P}_\tau$ be the collection of shapes $J \subset \mathbb{Z}/f\mathbb{Z}$ such that*

- *if $i-1 \in J$ and $i \notin J$, then $z_i \neq p-1$;*
- *if $i-1 \notin J$ and $i \in J$, then $z_i \neq 0$.*

(Recall z_i from (1.4.1)).

We denote by $\mathcal{Z}^\tau(J)$ the irreducible component of $\mathcal{Z}^\tau$ indexed by J. [CEGS19, Prop. 4.6.13] shows that $\mathcal{Z}^\tau(J)$ is the scheme-theoretic image of $\mathcal{C}^{\tau,\mathrm{BT}}(J)$. The irreducible components of $\mathcal{Z}$ are indexed by Serre weights, and for each σ a Serre weight, $\mathcal{Z}(\sigma)$ can show up in $\mathcal{Z}^\tau$ for multiple choices of τ. Thus we need to specify a dictionary to go from $J \in \mathcal{P}_\tau$ to a Serre weight σ.

For $J \in \mathcal{P}_\tau$, let δ_J denote the characteristic function of the set J. Define the integers b_i and a_i by

$$a_i = \begin{cases} z_i + \delta_{J^c}(i) & \text{if } i-1 \in J \\ 0 & \text{if } i-1 \notin J \end{cases}, \qquad b_i = \begin{cases} p-1-z_i-\delta_{J^c}(i) & \text{if } i-1 \in J \\ z_i - \delta_J(i) & \text{if } i-1 \notin J \end{cases}. \tag{4.1.1}$$

Viewing η' as a map $k^\times \to \mathbb{F}$ via Artin reciprocity, let σ_J be the Serre weight $\sigma_{\vec{a},\vec{b}} \otimes \eta' \circ \det$. Then by [CEGS19, Thm. 4.6.17, Appendix B], $\mathcal{Z}^\tau(J)$ is the irreducible component indexed by the Serre weight σ_J.

Proposition 4.1.2. *Set $J = \mathbb{Z}/f\mathbb{Z}$. Let σ be a Serre weight that is not a twist of either the trivial or the Steinberg representation. That is, $\sigma = \sigma_{\vec{a},\vec{b}}$ where $\vec{b} \notin \{(0,...,0),(p-1,...,p-1)\}$.*

Then we can find a unique principal series $\mathbb{F}$-type $\tau = \eta \oplus \eta'$ such that $\eta \neq \eta'$ and $\sigma = \sigma_J$.

Proof. Let $z_i = p-1-b_i$. Define η and η' via

$$\eta'(g) := \prod_{i=0}^{f-1}(\kappa_i \circ h(g)^{a_i - z_i}) \qquad\qquad \eta(g) := \eta'(g)\prod_{i=0}^{f-1}(\kappa_i \circ h(g)^{z_i}).$$

Let $\tau := \eta \oplus \eta'$. Clearly, $\sigma = \left(\otimes_{i=0}^{f-1}(\det^{z_i}\mathrm{Sym}^{b_i} k^2)\otimes_{k,\kappa_i}\mathbb{F}\right)\otimes \eta' \circ det = \sigma_J$ for inertial $\mathbb{F}$-type τ as desired. Any τ so chosen is unique by (4.1.1); $\vec{b}$ tells us exactly what the $\{z_i\}_i$ should be. Note that $\eta = \eta'$ if and only if all the z_i's are 0 or if all the z_i's are $p-1$. Both of these situations are ruled out by the hypotheses in the statement of the Proposition. □

Corollary 4.1.3. *Let S be the set of non-Steinberg Serre weights σ such that $\mathcal{Z}(\sigma)$ is the image of $\mathcal{C}^{\tau,\mathrm{BT}}(\mathbb{Z}/f\mathbb{Z})$ for some $\tau = \eta \oplus \eta'$ satisfying Assumption 3.0.1. Then $\sigma_{\vec{a},\vec{b}} \in S$ if and only if each of the following conditions are satisfied:*

1. *$\vec{b} \neq (0,0,\ldots,0)$,*
2. *$\vec{b} \neq (p-2,p-2,\ldots,p-2)$, and*
3. *Extend the indices of b_i's to all of $\mathbb{Z}$ by setting $b_{i+f} = b_i$. Then $(b_i)_{i\in\mathbb{Z}}$ does not contain a contiguous subsequence of the form $(0,p-2,\ldots,p-2,p-1)$, where the number of $p-2$'s in between 0 and $p-1$ can be anything in $\mathbb{Z}_{\geq 0}$.*

Proof. Proposition 4.1.2 accounts for the first condition. By (4.1.1), requiring τ to not face the first obstruction is equivalent to requiring $(b_i)_{i\in\mathbb{Z}}$ to not be made up entirely of concatenations of just two building blocks: $p-2$ and $(p-1,0)$. Similarly, requiring τ to not face the second obstruction is equivalent to requiring $(b_i)_{i\in\mathbb{Z}}$ to not contain a contiguous subsequence of the form $(0,p-2,\ldots,p-2,p-1)$ of length ≥ 2. If $(b_i)_{i\in\mathbb{Z}}$ is entirely made up of and contains both $p-2$ and $(p-1,0)$, then it automatically contains a contiguous subsequence of the form $(0,p-2,\ldots,p-2,p-1)$. Therefore, removing the redundant condition, we get the list of the conditions in the statement of the Corollary. □

4.2. **Presentations of components of $\mathcal{Z}$.** We will now show that if $\mathcal{Z}(\sigma)$ is as in the statement of Corollary 4.1.3, then it is isomorphic to $\mathcal{C}^{\tau,\mathrm{BT}}(\mathbb{Z}/f\mathbb{Z})$. A key ingredient in our proof will be the following proposition.

Proposition 4.2.1. *Fix $\tau = \eta \oplus \eta'$ a tame principal series $\mathbb{F}$-type with $\eta \neq \eta'$ such that τ does not face either the first or the second obstruction. Let $\sigma = \sigma_{\mathbb{Z}/f\mathbb{Z}}$. The map $q : [X/G] \cong \mathcal{X}(\tau) = \mathcal{C}^{\tau,\mathrm{BT}}(\mathbb{Z}/f\mathbb{Z}) \to \mathcal{Z}(\sigma)$ induced from Definition 2.2.4 is a monomorphism.*

The proof of this Proposition depends on the following Lemma.

Lemma 4.2.2. *Let R be an arbitrary $\mathbb{F}$-algebra. Let $\mathfrak{M}, \mathfrak{N} \in \mathcal{C}^{\tau,\mathrm{BT}}(R)$ such that with respect to some fixed inertial bases, the i-th Frobenius maps of $\mathfrak{M}$ and $\mathfrak{N}$ are in η-form. Upon restriction to the η'-eigenspace, suppose they are represented by*

$$F_i = \begin{pmatrix} a_i v & b_i \\ c_i v & d_i \end{pmatrix} \quad \text{and} \quad G_i = \begin{pmatrix} a_i' v & b_i' \\ c_i' v & d_i' \end{pmatrix}, \quad \text{respectively,} \tag{4.2.1}$$

where $a_i, a_i', b_i, b_i', c_i, c_i', d_i, d_i' \in R$ and $a_i d_i - b_i c_i$ and $a_i' d_i' - b_i' c_i'$ are units in R. Lastly, suppose that $\mathfrak{M}$ and $\mathfrak{N}$ are isomorphic as étale φ-modules, so that by (2.4.3), there exist $B_0, \dots, B_{f-1} \in \mathrm{GL}_2(R((u)))$ such that

$$G_i = B_i^{-1} F_i \left(Ad \begin{pmatrix} v^{p-1-z_i} & 0 \\ 0 & 1 \end{pmatrix} (\varphi(B_{i-1})) \right). \tag{4.2.2}$$

Then the following are true:

1. *$det(B_i) \in R[\![v]\!]^*$ and $vB_i \in M_2(R[\![v]\!])$.*
2. *If τ does not face the first obstruction, then $B_i \in \mathrm{GL}_2(R[\![v]\!])$.*

We observe that in the statement of Lemma 4.2.2, $B_i \in \mathrm{GL}_2(R((v)))$ by Lemma 2.1.5 and Definition 2.4.11.

Proof. (1) From (4.2.2), we see that

$$\det(B_i)\det(G_i) = \det(F_i)\varphi(\det(B_{i-1})).$$

Since $\mathrm{val}_v(\det(F_i)) = \mathrm{val}_v(\det(G_i)) = 1$, we have

$$\mathrm{val}_v(\det(B_i)) = \mathrm{val}_v(\varphi(\det(B_{i-1})) = p\mathrm{val}_v(\det(B_{i-1})).$$

Iterating this equation gives us

$$\mathrm{val}_v(\det(B_i)) = p\mathrm{val}_v(\det(B_{i-1})) = p^2\mathrm{val}_v(\det(B_{i-2})) = \cdots = p^f\mathrm{val}_v(\det(B_i)),$$

which shows $\mathrm{val}_v(\det(B_i)) = 0$. We now choose $k_i \in \mathbb{Z}_{\geq 0}$ minimal such that

$$B_i = v^{-k_i} B_i^+ = v^{-k_i} \begin{pmatrix} s_1^{(i)} & s_2^{(i)} \\ s_3^{(i)} & s_4^{(i)} \end{pmatrix}, \qquad \text{where } s_1^{(i)}, s_2^{(i)}, s_3^{(i)}, s_4^{(i)} \in R[\![v]\!].$$

Then from (4.2.2), we have

$$\begin{aligned} vF_i^{-1} B_i^+ G_i &= v^{k_i - pk_{i-1}+1} \begin{pmatrix} v^{p-1-z_i} & 0 \\ 0 & 1 \end{pmatrix} \varphi(B_{i-1}^+) \begin{pmatrix} v^{-p+1+z_i} & 0 \\ 0 & 1 \end{pmatrix} \\ &= v^{k_i - pk_{i-1}+1} \begin{pmatrix} \varphi(s_1^{(i-1)}) & v^{p-1-z_i}\varphi(s_2^{(i-1)}) \\ v^{-p+z_i+1}\varphi(s_3^{(i-1)}) & \varphi(s_4^{(i-1)}) \end{pmatrix}. \end{aligned}$$

Since $\det(F_i) \in vR[\![v]\!]^*$, we know $vF_i^{-1} \in \mathrm{M}_2(R[\![v]\!])$. Since k_{i-1} is chosen to be minimal, we must have $\mathrm{val}_v(\varphi(s_{m_i}^{(i-1)})) = 0$ for some $m_i \in \{1,2,3,4\}$. We have:

$$\begin{aligned}
m_i \in \{1,4\} &\implies 0 \le k_i - pk_{i-1} + 1 &&\implies k_i \ge pk_{i-1} - 1,\\
m_i = 2 &\implies 0 \le k_i - pk_{i-1} + 1 + p - 1 - z_i &&\implies k_i \ge pk_{i-1} - 1 - (p-1-z_i),\\
m_i = 3 &\implies 0 \le k_i - pk_{i-1} + 1 - p + z_i + 1 &&\implies k_i \ge pk_{i-1} - 1 + (p - z_i - 1).
\end{aligned}$$

In other words, $k_i \ge pk_{i-1} - 1 - \epsilon_i$ where

$$\epsilon_i = \begin{cases} 0 & \text{if } m_i \in \{1,4\} \\ p-1-z_i & \text{if } m_i = 2 \\ -(p-1-z_i) & \text{if } m_i = 3 \end{cases}$$

Iterating, we get

$$\begin{aligned}
&k_i \ge p^f k_i - (p^{f-1} + p^{f-2} + \dots + 1) - (p^{f-1}\epsilon_i + p^{f-2}\epsilon_{i-1} + \dots + \epsilon_{i-f+1})\\
\iff\ &(p^f - 1)k_i \le (p^{f-1} + p^{f-2} + \dots + 1) + (p^{f-1}\epsilon_i + p^{f-2}\epsilon_{i-1} + \dots + \epsilon_{i-f+1}).
\end{aligned}$$

Since $(p^{f-1}\epsilon_i + p^{f-2}\epsilon_{i-1} + \dots + \epsilon_{i-f+1})$ is bounded above by $(p-1)(p^{f-1} + p^{f-2} + \dots + 1) = p^f - 1$, we must have $k_i \in \{0,1\}$, showing that $B_i \in vM_2(R[\![v]\!])$.

(2) Evidently, $k_i = 1$ implies that $(p^{f-1}\epsilon_i + p^{f-2}\epsilon_{i-1} + \dots + \epsilon_{i-f+1}) \ge (p-2)(p^{f-1} + p^{f-2} + \dots + 1)$. This further implies that $\epsilon_i \ge p-2$, or equivalently, $z_i \in \{0,1\}$ and v divides $s_1^{(i-1)}$, $s_3^{(i-1)}$, and $s_4^{(i-1)}$.

From the entries of the matrices in (4.2.2), we get the following equalities:

$$v^{k_i+pk_{i-1}} a_i' \det(B_i) = \begin{array}{c} a_i\varphi(s_1^{(i-1)})s_4^{(i)} + v^{-p+z_i} b_i \varphi(s_3^{(i-1)})s_4^{(i)} \\ -c_i\varphi(s_1^{(i-1)})s_2^{(i)} - v^{-p+z_i} d_i\varphi(s_3^{(i-1)})s_2^{(i)} \end{array}, \tag{4.2.3}$$

$$v^{k_i+pk_{i-1}} b_i' \det(B_i) = \begin{array}{c} v^{p-z_i} a_i\varphi(s_2^{(i-1)})s_4^{(i)} + b_i \varphi(s_4^{(i-1)})s_4^{(i)} \\ -v^{p-z_i}c_i\varphi(s_2^{(i-1)})s_2^{(i)} - d_i\varphi(s_4^{(i-1)})s_2^{(i)} \end{array}, \tag{4.2.4}$$

$$v^{k_i+pk_{i-1}} c_i' \det(B_i) = \begin{array}{c} -a_i\varphi(s_1^{(i-1)})s_3^{(i)} - v^{-p+z_i} b_i \varphi(s_3^{(i-1)})s_3^{(i)} \\ +c_i\varphi(s_1^{(i-1)})s_1^{(i)} + v^{-p+z_i} d_i\varphi(s_3^{(i-1)})s_1^{(i)} \end{array}, \tag{4.2.5}$$

$$v^{k_i+pk_{i-1}} d_i' \det(B_i) = \begin{array}{c} -v^{p-z_i} a_i\varphi(s_2^{(i-1)})s_3^{(i)} - b_i \varphi(s_4^{(i-1)})s_3^{(i)} \\ +v^{p-z_i}c_i\varphi(s_2^{(i-1)})s_1^{(i)} + d_i\varphi(s_4^{(i-1)})s_1^{(i)} \end{array}. \tag{4.2.6}$$

We claim that if there exists a j with $k_{j-1} = 0$, then k_j equals 0 as well. We show this by contradiction, assuming that $k_j = 1$.

Let $i = j$ in (4.2.4). If $b_j' \neq 0$, the v-adic valuation of the LHS in (4.2.4) is 1, while that of the RHS is $\ge p - z_i$ (since $s_4^{(j-1)}$ is divisible by v). This gives a contradiction, forcing $b'_j = 0$. Since $a'_jd'_j - b'_jc'_j \in R^*$, $d'_j \neq 0$. Comparing v-adic valuations of LHS and RHS in (4.2.6) gives a contradiction.

Therefore, if there exists $j \in \mathbb{Z}/f\mathbb{Z}$ such that $k_j = 0$, then $k_i = 0$ for all $i \in \mathbb{Z}/f\mathbb{Z}$. Now we show that there must exist a j such that $k_j = 0$. Suppose there doesn't and each $k_i = 1$. Therefore, each $z_i \in \{0, 1\}$ and v divides each $s_1^{(i)}$, $s_3^{(i)}$ and $s_4^{(i)}$, but does not divide $s_2^{(i)}$. Moreover, because v does not divide $\det(B_i)$, v^2 must divide $s_3^{(i)}$. Since τ does not face the first obstruction, there exists some l such that $z_l = 0$. Taking $i = l$, consider the equations $s_1^{(l)}$(4.2.3) $+s_2^{(l)}$(4.2.5) and $s_1^{(l)}$(4.2.4) $+s_2^{(l)}$(4.2.6). Dividing both the equations by $\det(B_l^+) = v^2\det(B_l)$, we obtain:

$$v^{p-1}(s_1^{(l)}a'_l + s_2^{(l)}c'_l) = a_l\varphi(s_1^{(l-1)}) + v^{-p}b_l\varphi(s_3^{(l-1)}), \tag{4.2.7}$$

$$v^{p-1}(s_1^{(l)}b'_l + s_2^{(l)}d'_l) = v^p a_l\varphi(s_2^{(l-1)}) + b_l\varphi(s_4^{(l-1)}). \tag{4.2.8}$$

For each of the above equations, the RHS is divisible by v^p. Therefore, the same is true for LHS. Since v does not divide $s_2^{(l)}$, c'_l and d'_l must be 0, contradicting the fact that $a'_ld'_l - b'_lc'_l$ is a unit in R. This implies that either k_l or k_{l+1} is 0. Therefore, all k_i equal 0, and hence, $B_i \in M_2(R[\![v]\!])$ for all i.

Similar considerations applied to $\{B_i^{-1}\}_i$ show that each B_i is in $GL_2(R[\![v]\!])$.

□

Proof of Proposition 4.2.1. Let R be an arbitrary $\mathbb{F}$-algebra, and let $\mathfrak{M}, \mathfrak{N} \in \mathcal{X}(\tau)(R) \cong [X/G](R)$ be two Breuil-Kisin modules that become isomorphic after inverting u. By Proposition 3.1.4, the Frobenius matrices of $\mathfrak{M}$ and $\mathfrak{N}$ can be written in the form described in the statement of Lemma 4.2.2 (after passing to an affine cover of Spec R if necessary). Denoting the Frobenius matrices of $\mathfrak{M}$ by $\{F_i\}_i$ and those of $\mathfrak{N}$ by $\{G_i\}_i$, the isomorphism between $q(\mathfrak{M})$ and $q(\mathfrak{N})$ is described by invertible matrices $\{B_i\}_i$ satisfying (4.2.2). Lemma 4.2.2 shows each $B_i \in \mathrm{GL}_2(R[\![v]\!])$; we further claim that each B_i is upper triangular mod v. If this is true, then by comparison with the form of inertial base change matrices for η'-eigenspace, the set $\{B_i\}_i$ gives an isomorphism between $\mathfrak{M}$ and $\mathfrak{N}$. Therefore, $X \times_{\mathcal{X}(\tau)} X \to X \times_{\mathcal{Z}} X$ is an isomorphism. Using 3.1.6, the diagonal of q is an isomorphism (the argument for this is the same as in the first paragraph of the proof of Lemma 3.1.9). Therefore, q is a monomorphism.

To justify the claim, we continue using the notation of the proof of Lemma 4.2.2. Since each k_i is 0, we obtain the following equation upon dividing $s_1^{(i)}$(4.2.3) $+s_2^{(i)}$(4.2.5) by $\det(B_i)$.

$$s_1^{(i)}a'_i + s_2^{(i)}c'_i = a_i\varphi(s_1^{(i-1)}) + v^{-p+z_i}b_i\varphi(s_3^{(i-1)}) \tag{4.2.9}$$

Since the LHS is integral, the same must be true for RHS. Therefore if $s_3^{(i-1)}$ is not divisible by v, we must have $b_i = 0$. Plugging $b_i = 0$ into (4.2.3) and (4.2.5), we must have either $d_i = 0$ or $s_1^{(i)} \equiv s_2^{(i)} \equiv 0 \mod v$. The former is impossible since $a_id_i - b_ic_i$ is nonzero, while the latter is impossible because B_i is invertible. Consequently, each $s_3^{(i-1)}$ is divisible by v and the claim rests. □

Corollary 4.2.3. *Fix $\tau = \eta \oplus \eta'$ a tame principal series $\mathbb{F}$-type with $\eta \neq \eta'$ such that τ does not face either the first or the second obstruction. Let $\sigma = \sigma_{\mathbb{Z}/f\mathbb{Z}}$. The map $q : [X/G] \cong \mathcal{C}^{\tau,\mathrm{BT}}(\mathbb{Z}/f\mathbb{Z}) \to \mathcal{Z}(\sigma)$ is an isomorphism.*

Proof. The map q is proper by [CEGS19, Thm. 3.9.2] and a monomorphism by Proposition 4.2.1. Since q is scheme-theoretically dominant and proper monomorphisms are the same as closed immersions (by [Sta18, Tag 0418, Tag 04XV]), q must be an isomorphism. □

5. Conclusion

Theorem 5.0.1. *Let $p > 2$. Let K be an unramified extension of $\mathbb{Q}_p$ of degree f with residue field k. Let $\mathcal{Z}(\sigma)$ be the irreducible component of $\mathcal{Z}$ indexed by a non-Steinberg Serre weight $\sigma = \sigma_{\vec{a},\vec{b}} = \bigotimes_{i=0}^{f-1}(\det^{a_i}\operatorname{Sym}^{b_i} k^2) \otimes_{k,\kappa_i} \mathbb{F}$ satisfying the following properties:*

(1) $\vec{b} \neq (0,0,\dots,0)$,
(2) $\vec{b} \neq (p-2, p-2, \dots, p-2)$,
(3) *Extend the indices of b_i's to all of $\mathbb{Z}$ by setting $b_{i+f} = b_i$. Then $(b_i)_{i\in\mathbb{Z}}$ does not contain a contiguous subsequence of the form $(0, p-2, \dots, p-2, p-1)$ of length ≥ 2.*

Then $\mathcal{Z}(\sigma)$ is smooth and isomorphic to a quotient of $\mathrm{GL}_2 \times \mathrm{SL}_2^{f-1}$ by $\mathbb{G}_m^{f+1} \times \mathbb{G}_a^f$. The ring of global functions of $\mathcal{Z}(\sigma)$ is isomorphic to $\mathbb{F}[x,y][\frac{1}{y}]$.

Proof. Follows directly from Corollaries 3.2.5, 4.1.3 and 4.2.3. □

Remark 5.0.2. *In fact, when $f \geq 2$ and $p > 3$, we can allow $\vec{b} = (0,0,\dots,0)$ (see discussion in Section A.4).*

Appendix A. Allowing η'-forms

The objective of this Appendix is to show that allowing some of the Frobenius matrices to be in η'-form does not allow us to obtain information on more irreducible components, with the exception of the component indexed by the trivial Serre weight. Before we embark on a proof, we first survey the overall strategy employed in the main body of the paper, and analyze how it might be affected by allowing some Frobenius matrices to be in η'-form.

A key ingredient in the proof of our main theorem is constructing the functor $\widetilde{\mathcal{T}} : [X/G] \to \mathcal{X}(\tau)$ (see Definition 3.1.3), where $X = \mathrm{GL}_2 \times \mathrm{SL}_2^{f-1}$ and $G = \mathbb{G}_m^{f+1} \times U^f$, and then showing that it is an isomorphism (see Proposition 3.1.4). The proof of the isomorphism relies, among other things, on the following:

(1) Let $\mathfrak{M}$ be a regular Breuil-Kisin module and let $\{F_i\}_i$ be the set of its Frobenius matrices with respect to some choice of inertial bases. Suppose that each F_i is in

η-form. Then, upon imposing some conditions on $(z_i)_i$, we can guarantee that $\mathfrak{M}$ is not of bad genre and therefore the algorithm in Proposition 2.3.7 converges to give Frobenius matrices in CDM form. The minimal set of values of $(z_i)_i$ we need to exclude constitutes the definition of the first obstruction.

(2) For $\mathfrak{M}$ as above, we also need to obtain the CDM form of Frobenius matrices through an action of G. The conditions on $(z_i)_i$ that prohibit this constitute the definition of the second obstruction.

After showing that $\widetilde{\mathcal{T}} : [X/G] \to \mathcal{X}(\tau)$ is an isomorphism, our next step is to identify the irreducible component $\mathcal{X}(\tau) \subset \mathcal{C}^{\tau,\mathrm{BT}}$ by its shape index. We identify this shape index to be $\mathbb{Z}/f\mathbb{Z}$ by observing that $\mathcal{C}^{\tau,\mathrm{BT}}(\mathbb{Z}/f\mathbb{Z})$ is the only irreducible component containing a dense set of points with each Frobenius map of genre I_η (see Lemma 3.2.3). Using (4.1.1), we finally compute the Serre weight index of $\mathcal{Z}^\tau(\mathbb{Z}/f\mathbb{Z})$ which is the image of $\mathcal{C}^{\tau,\mathrm{BT}}(\mathbb{Z}/f\mathbb{Z})$ in $\mathcal{Z}$.

If we allow η'-forms, we will need to change the definitions of first and second obstructions since they are presently tailored to work in the situation where each Frobenius matrix is in η-form. Furthermore, the definition of $\mathcal{T}$ (and therefore of $\tilde{\mathcal{T}}$) will have to be modified to allow for the image to have some Frobenius maps in η'-form. The image of $\widetilde{\mathcal{T}}$ will no longer be $\mathcal{C}^{\tau,\mathrm{BT}}(\mathbb{Z}/f\mathbb{Z})$. We will need to compute the correct shape index as a function of the indices $i \in \mathbb{Z}/f\mathbb{Z}$ for which we are allowing η'-form Frobenius matrices, and then compute the Serre weight index using the correct shape index.

Instead of directly replicating the structure of our proofs in the main body of the text, we will now evaluate the effect of allowing η'-form Frobenius matrices in a slightly non-linear fashion. We will first compute the shape J needed such that $\mathcal{C}^{\tau,\mathrm{BT}}(J)$ contains a dense set of points with some Frobenius maps of genre I_η and others of genre $\mathrm{I}_{\eta'}$ as well as investigate the relationship of J to Serre weights. Next, we will compute the altered conditions for first and second obstructions. Finally, we will show that although we could not include twists of trivial Serre weight in our main analysis, we can include them if we allow η'-form Frobenius matrices, and that this is the only extra advantage to be gained by allowing η'-form matrices.

To start, we introduce some notation:

We let $T \subset \mathbb{Z}/f\mathbb{Z}$ be the fixed set of indices i such that the i-th Frobenius map is in η-form, while T^c is the set of indices i such that the i-th Frobenius map is in η'-form.

Definition A.0.1. *Let $i \in \mathbb{Z}/f\mathbb{Z}$. We say that $(i-1, i)$ is a transition if one of $\{i-1, i\}$ is in T and the other in T^c.*

Given $\tau = \eta \oplus \eta'$ with $\eta \neq \eta'$, define $(\tilde{z}_i)_i$ via:

$$\tilde{z}_i = \begin{cases} z_i & \text{if } i-1 \in T, i \in T, \\ z_i + 1 & \text{if } i-1 \in T, i \notin T, \\ p - z_i & \text{if } i-1 \notin T, i \in T, \\ p-1-z_i & \text{if } i-1 \notin T, i \notin T, \end{cases} \tag{A.0.1}$$

where z_i is defined in (1.4.1). As with z_i, we will take the indexing set of $\tilde{z}_i$ to be either $\mathbb{Z}/f\mathbb{Z}$ or $\mathbb{Z}$ depending on the situation.

Remark A.0.2. *By (A.0.1), $\tilde{z}_i \neq 0$ whenever $(i-1, i)$ is a transition.*

A.1. **Shapes.**

Lemma A.1.1. *Let τ be a tame principal series $\mathbb{F}$-type. Suppose $\mathcal{C}^{\tau,\mathrm{BT}}(J)$ is an irreducible component of $\mathcal{C}^{\tau,\mathrm{BT}}$ comprising a dense set of $\overline{\mathbb{F}}_p$-points corresponding to Breuil-Kisin modules that satisfy the following:*

- *The genre of the i-th Frobenius map is I_η for $i \in T$.*
- *The genre of the i-th Frobenius map is $\mathrm{I}_{\eta'}$ for $i \notin T$.*

Then $J = T$.

Proof. By the argument in the proof of Lemma 3.2.3, $\mathcal{C}^{\tau,\mathrm{BT}}(J)$ contains a dense constructible set of points such that if $i \in J$, then the upper left entry of i-th Frobenius is 0 or v-divisible, making it necessarily of genre I_η or II. On the other hand, if $i \notin J$, then the lower right entry of i-th Frobenius is either 0 or v-divisible, making it necessarily of genre $\mathrm{I}_{\eta'}$ or II. □

Lemma A.1.2. *Let $\mathcal{C}^{\tau,\mathrm{BT}}(J)$ be as in Lemma A.1.1. Then $\mathcal{C}^{\tau,\mathrm{BT}}(J)$ is a cover of an irreducible component of $\mathcal{Z}$ if and only if $J \in \mathcal{P}_\tau$ if and only if for each i, $\tilde{z}_i \neq p$.*

Proof. $\mathcal{C}^{\tau,\mathrm{BT}}(J)$ is a cover of an irreducible component of $\mathcal{Z}$ if and only if $J \in \mathcal{P}_\tau$ by [CEGS19, Thm. 4.6.12]. By the definition of $\mathcal{P}_\tau$ (Definition 4.1.1) and the fact that $J = T$, the condition on $(\tilde{z}_i)_i$ is immediate. □

Since the strategy of this paper rests on covering a suitable irreducible component of $\mathcal{Z}$ by the irreducible component of $\mathcal{C}^{\tau,\mathrm{BT}}$ in the image of $\mathcal{T}$, it is reasonable to impose the condition that for each i, $\tilde{z}_i \neq p$.

Remark A.1.3. *Suppose $J = T$ as in Lemma A.1.1 and $\tilde{z}_i \neq p$. Since $J \in \mathcal{P}_\tau$, we may compute the Serre weight corresponding to J. By (4.1.1), the symmetric powers of the Serre weight are given by $b_i = p - 1 - \tilde{z}_i$.*

A.2. **First obstruction.** As in the greater part of Section 2.3, we will assume that all Breuil-Kisin modules in this section are regular (see Definition 2.3.4). We will also assume that $\tilde{z}_i \neq p$.

Definition A.2.1. *Let $\mathfrak{M}$ be a Breuil-Kisin module over an $\mathbb{F}$-algebra R with Frobenius matrices $\{F_i\}_i$ written with respect to some inertial bases. We say that $\mathcal{G}(\mathfrak{M}_i) = \mathcal{G}(F_i) = \mathrm{I}$ if $\mathcal{G}(\mathfrak{M}_i) = \mathcal{G}(F_i) \in \{\mathrm{I}_\eta, \mathrm{I}_{\eta'}\}$.*

Lemma A.2.2. *Let R be an Artinian local ring over $\mathbb{F}$ with maximal ideal $\mathfrak{m}$. A regular Breuil-Kisin module $\mathfrak{M}$ defined over R is of bad genre if and only if the following conditions are satisfied (assuming $\tilde{z}_i \neq p$ for all i):*

(1) *If $(i-1, i)$ is not a transition, then $(\mathcal{G}(F_i), \tilde{z}_i) \in \{(\mathrm{II}, 0), (\mathrm{II}, p-1), (\mathrm{I}, 1), (\mathrm{I}, p-1)\}$.*
If $(i-1, i)$ is a transition, then $(\mathcal{G}(F_i), \tilde{z}_i) \in \{(\mathrm{II}, 1), (\mathrm{I}, 1), (\mathrm{I}, p-1)\}$.
(2) *If $(i-1, i)$ is not a transition and $(\mathcal{G}(F_i), \tilde{z}_i) = (\mathrm{II}, 0)$, then*
$(\mathcal{G}(F_{i+1}), \tilde{z}_{i+1}) = (\mathrm{I}, p-1)$, or $(\mathcal{G}(F_{i+1}), \tilde{z}_{i+1}) = (\mathrm{II}, p-1)$ with $(i, i+1)$ not a transition.
(3) *If $(i-1, i)$ is not a transition and $(\mathcal{G}(F_i), \tilde{z}_i) \in \{(\mathrm{II}, p-1), (\mathrm{I}, 1), (\mathrm{I}, p-1)\}$, then $(\mathcal{G}(F_{i+1}), \tilde{z}_{i+1}) = (\mathrm{II}, 0)$ with $(i, i+1)$ not a transition, or $(\mathcal{G}(F_{i+1}), \tilde{z}_{i+1}) = (\mathrm{II}, 1)$ with $(i, i+1)$ a transition, or $(\mathcal{G}(F_{i+1}), \tilde{z}_{i+1}) = (\mathrm{I}, 1)$.*
(4) *If $(i-1, i)$ is a transition and $(\mathcal{G}(F_i), \tilde{z}_i) \in \{(\mathrm{II}, 1), (\mathrm{I}, 1), (\mathrm{I}, p-1)\}$, then $(\mathcal{G}(F_{i+1}), \tilde{z}_{i+1}) = (\mathrm{II}, 0)$ with $(i, i+1)$ not a transition, or $(\mathcal{G}(F_{i+1}), \tilde{z}_{i+1}) = (\mathrm{II}, 1)$ with $(i, i+1)$ a transition, or $(\mathcal{G}(F_{i+1}), \tilde{z}_{i+1}) = (\mathrm{I}, 1)$.*

Proof. Suppose $i \in T$. We restate the conditions for bad genre by expressing the conditions from Definition 2.3.6 in terms of $\tilde{z}_i$:

(1) If $i-1 \in T$, then $(\mathcal{G}(F_i), \tilde{z}_i) \in \{(\mathrm{II}, 0), (\mathrm{II}, p-1), (\mathrm{I}, 1), (\mathrm{I}, p-1)\}$.
If $i-1 \notin T$, then $(\mathcal{G}(F_i), \tilde{z}_i) \in \{(\mathrm{II}, 1), (\mathrm{I}, 1), (\mathrm{I}, p-1)\}$.
(2) If $i-1 \in T$ and $(\mathcal{G}(F_i), \tilde{z}_i) = (\mathrm{II}, 0)$, then
$(\mathcal{G}(F_{i+1}), \tilde{z}_{i+1}) = (\mathrm{II}, p-1)$ with $i+1 \in T$, or $(\mathcal{G}(F_{i+1}), \tilde{z}_{i+1}) = (\mathrm{I}, p-1)$.
(3) If $i-1 \in T$ and $(\mathcal{G}(F_i), \tilde{z}_i) \in \{(\mathrm{II}, p-1), (\mathrm{I}, 1), (\mathrm{I}, p-1)\}$, then $(\mathcal{G}(F_{i+1}), \tilde{z}_{i+1}) = (\mathrm{II}, 0)$ with $i+1 \in T$ or $(\mathcal{G}(F_{i+1}), \tilde{z}_{i+1}) = (\mathrm{II}, 1)$ with $i+1 \notin T$ or $(\mathcal{G}(F_{i+1}), \tilde{z}_{i+1}) = (\mathrm{I}, 1)$.
(4) If $i-1 \notin T$ and $(\mathcal{G}(F_i), \tilde{z}_i) \in \{(\mathrm{II}, 1), (\mathrm{I}, 1), (\mathrm{I}, p-1)\}$, then $(\mathcal{G}(F_{i+1}), \tilde{z}_{i+1}) = (\mathrm{II}, 0)$ with $i+1 \in T$ or $(\mathcal{G}(F_{i+1}), \tilde{z}_{i+1}) = (\mathrm{II}, 1)$ with $i+1 \notin T$ or $(\mathcal{G}(F_{i+1}), \tilde{z}_{i+1}) = (\mathrm{I}, 1)$.

By symmetry, for $i \notin T$, the conditions for bad genre are:

(1) If $i-1 \notin T$, then $(\mathcal{G}(F_i), \tilde{z}_i) \in \{(\mathrm{II}, 0), (\mathrm{II}, p-1), (\mathrm{I}, 1), (\mathrm{I}, p-1)\}$.
If $i-1 \in T$, then $(\mathcal{G}(F_i), \tilde{z}_i) \in \{(\mathrm{II}, 1), (\mathrm{I}, 1), (\mathrm{I}, p-1)\}$.
(2) If $i-1 \notin T$ and $(\mathcal{G}(F_i), \tilde{z}_i) = (\mathrm{II}, 0)$, then
$(\mathcal{G}(F_{i+1}), \tilde{z}_{i+1}) = (\mathrm{II}, p-1)$ with $i+1 \notin T$, or $(\mathcal{G}(F_{i+1}), \tilde{z}_{i+1}) = (\mathrm{I}, p-1)$.

(3) If $i-1 \notin T$ and $(\mathcal{G}(F_i), \tilde{z}_i) \in \{(\mathrm{II}, p-1), (\mathrm{I}, 1), (\mathrm{I}, p-1)\}$, then $(\mathcal{G}(F_{i+1}), \tilde{z}_{i+1}) = (\mathrm{II}, 0)$ with $i+1 \notin T$ or $(\mathcal{G}(F_{i+1}), \tilde{z}_{i+1}) = (\mathrm{II}, 1)$ with $i+1 \in T$ or $(\mathcal{G}(F_{i+1}), \tilde{z}_{i+1}) = (\mathrm{I}, 1)$.

(4) If $i-1 \in T$ and $(\mathcal{G}(F_i), \tilde{z}_i) \in \{(\mathrm{II}, 1), (\mathrm{I}, 1), (\mathrm{I}, p-1)\}$, then $(\mathcal{G}(F_{i+1}), \tilde{z}_{i+1}) = (\mathrm{II}, 0)$ with $i+1 \notin T$ or $(\mathcal{G}(F_{i+1}), \tilde{z}_{i+1}) = (\mathrm{II}, 1)$ with $i+1 \in T$ or $(\mathcal{G}(F_{i+1}), \tilde{z}_{i+1}) = (\mathrm{I}, 1)$.

Bringing the two sets of conditions together, the conditions for bad genre are as in the statement of the lemma. □

From Lemma A.2.2, it is immediate that the following is the appropriate generalization of the definition of first obstruction.

Definition A.2.3. *We say that a tame prinicipal series $\mathbb{F}$-type τ faces the first obstruction if $(\tilde{z}_i)_{i\in\mathbb{Z}}$ is made up entirely of the building blocks 1 and $(0, p-1)$.*

A.3. **Second obstruction.** To compute the right form of second obstruction conditions, we first state a version of Lemma 2.4.9 for Frobenius matrices in η'-form.

Lemma A.3.1. *Let R be an Artinian local ring over $\mathbb{F}$ with maximal ideal $\mathfrak{m}$. Let $\mathfrak{M}$ be a regular Breuil-Kisin module, not of bad genre. Suppose with respect to an inertial basis, F_i has the form*

$$\begin{pmatrix} a_i & u^{e-\gamma_i} b_i \\ u^{\gamma_i} c_i & v d_i \end{pmatrix}$$

with $a_i, b_i, c_i, d_i \in R$. Let

$$P^{(j)} = \lim_{n\to\infty} P_{j+nf} = \begin{pmatrix} q_j & u^{e-\gamma_j} r_j \\ u^{\gamma_j} s_j & t_j \end{pmatrix}$$

denote the base change matrices described in the proof of Proposition 2.3.7. Let $F_i' = (P^{(i+1)})^{-1} F_i \varphi(P_i)$ be the matrix in 2.3.2, and explicitly, let

$$F_i' = \begin{pmatrix} a_i' & b_i' u^{e-\gamma_i} \\ c_i' u^{\gamma_i} & v d_i' \end{pmatrix}.$$

For any $\sigma \in R[\![v]\!]$, denote by $\overline{\sigma}$ the constant part of σ.

Then

$$F_i' = \begin{cases} Ad\begin{pmatrix} \frac{b_i+a_i\overline{r_{i-1}}}{b_i} & 0 \\ 0 & 1 \end{pmatrix}\left(\begin{pmatrix} a_i & u^{e-\gamma_i}b_i \\ 0 & (d_i-\frac{b_i}{a_i}c_i)v \end{pmatrix}\right) & \text{if } \mathcal{G}(F_i)=I_{\eta'}, z_i=p-1, \\ \begin{pmatrix} a_i & u^{e-\gamma_i}b_i \\ 0 & (d_i-\frac{b_i}{a_i}c_i)v \end{pmatrix} & \text{if } \mathcal{G}(F_i)=I_{\eta'}, z_i\neq p-1, \\ Ad\begin{pmatrix} \frac{b_i+a_i\overline{r_{i-1}}}{b_i} & 0 \\ 0 & 1 \end{pmatrix}\left(\begin{pmatrix} a_i & u^{e-\gamma_i}b_i \\ u^{\gamma_i}(c_i-\frac{a_i}{b_i}d_i) & 0 \end{pmatrix}\right) & \text{if } \mathcal{G}(F_i)=\mathrm{II}, z_i=p-1, \\ \begin{pmatrix} a_i & u^{e-\gamma_i}b_i \\ u^{\gamma_i}(c_i-\frac{a_i}{b_i}d_i) & 0 \end{pmatrix} & \text{if } \mathcal{G}(F_i)=\mathrm{II}, z_i\neq p-1, \end{cases}$$

where $Ad\, M\,(N)$ denotes the matrix MNM^{-1}.

Proof. By Lemma 2.4.9 using symmetry. ☐

Analogous to Proposition 2.4.8, we define a left action of lower unipotent matrices on η'-form via:

$$\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \star \begin{pmatrix} a_i & u^{e-\gamma_i}b_i \\ u^{\gamma_i}c_i & vd_i \end{pmatrix} = \begin{pmatrix} a_i & u^{e-\gamma_i}b_i \\ u^{\gamma_i}(c_i+ya_i) & v(d_i+yb_i) \end{pmatrix}. \tag{A.3.1}$$

We will assume now that $\mathfrak{M}$ is a regular Breuil-Kisin module with Frobenius matrices $\{F_i\}_i$ such that for $i \in T$, $F_i = \left(\begin{smallmatrix} va_i & u^{e-\gamma_i}b_i \\ u^{\gamma_i}c_i & d_i \end{smallmatrix}\right)$ and for $i \notin T$, $F_i = \left(\begin{smallmatrix} a_i & u^{e-\gamma_i}b_i \\ u^{\gamma_i}c_i & vd_i \end{smallmatrix}\right)$ with $a_i, b_i, c_i, d_i \in R$. Our objective is to find the minimal set of conditions on z_i that prohibit unipotent action (upper or lower, depending on the form of F_i) from giving F_i' (F_i' are as defined in Lemmas 2.4.9 and A.3.1). Evidently, left unipotent action on F_i fails to give F_i' if and only if one of the following is true:

- $i \in T$, $z_i = 0$ and $s_{i-1} \not\equiv 0 \mod v$, or
- $i \notin T$, $z_i = p-1$ and $r_{i-1} \not\equiv 0 \mod v$.

Recall that $P^{(i)} = \mathcal{B}(F_i\varphi(P^{(i-1)}))(\Delta^i)^{-1} = \mathcal{B}(F_i)\mathcal{B}(M_i\varphi(P^{(i-1)}))(\Delta^i)^{-1}$. Also by the explicit calculations in Lemma 2.4.9, $\mathcal{B}(F_i)$ is upper triangular if $i \in T$ and correspondingly, $\mathcal{B}(F_i)$ is lower triangular if $i \notin T$.

We want to now ascertain criteria for when $s_{i-1} \not\equiv 0 \bmod v$. We have the following possibilities:

(1) If $i-1 \in T$, $\mathcal{B}(F_{i-1})$ is upper triangular. Therefore, $s_{i-1} \not\equiv 0$ if and only if $\mathcal{B}(M_{i-1}\varphi(P^{(i-2)}))$ is not upper triangular mod $u^e R[\![u]\!]$. By the calculations in Lemma 2.3.12, this can happen only if one of the following statements holds:
   (a) $z_{i-1} = 1$ and $s_{i-2} \not\equiv 0$ mod v. In this situation, s_{i-1} is a multiple of s_{i-2} mod v.
   (b) $z_{i-1} = p-1$ and $r_{i-2} \not\equiv 0$ mod v. In this situation, s_{i-1} is a multiple of r_{i-2} mod v.

(2) If $i-1 \notin T$, $\mathcal{B}(F_{i-1})$ is lower triangular. In this case, if $\mathcal{G}(F_{i-1}) = \mathrm{I}_{\eta'}$, $s_{i-1} \equiv Cc_{i-1}$ mod v where $C \in R^*$. If $\mathcal{G}(F_{i-1}) = \mathrm{II}$, s_{i-1} is an R-linear combination of r_{i-2} and d_{i-1} mod v.

Similarly, for the situation where $r_{i-1} \not\equiv 0$ mod v, we have the following possibilities:

(1) If $i-1 \notin T$, $\mathcal{B}(F_{i-1})$ is lower triangular. Therefore, $r_{i-1} \not\equiv 0$ if and only if $\mathcal{B}(M_{i-1}\varphi(P^{(i-2)}))$ is not lower triangular mod $u^e R[\![u]\!]$. By the calculations in Lemma 2.3.12, this can happen only if one of the following statements holds:
   (a) $z_{i-1} = p-2$ and $r_{i-2} \not\equiv 0$ mod v. In this situation, r_{i-1} is a multiple of r_{i-2} mod v.
   (b) $z_{i-1} = 0$ and $s_{i-2} \not\equiv 0$ mod v. In this situation, r_{i-1} is a multiple of s_{i-2} mod v.

(2) If $i-1 \in T$, $\mathcal{B}(F_{i-1})$ is upper triangular. In this case, if $\mathcal{G}(F_{i-1}) = \mathrm{I}_{\eta}$, $r_{i-1} \equiv Cb_i$ mod v where $C \in R^*$. If $\mathcal{G}(F_{i-1}) = \mathrm{II}$, r_{i-1} is an R-linear combination of s_{i-2} and a_{i-1} mod v.

Suppose $s_{i-1} \not\equiv 0$ mod v. Then z_i is preceded by some sequence $(z_{i-k-1}, ..., z_{i-1}) = (1, ..., 1)$ with $k \geq -1$ and such that $[i-k-2, i-2] \subset T$. When $k = -1$, we mean that the sequence is empty. This sequence of 1's must be preceded by either of the following:

- $z_{i-k-2} = p-1$ with $i-k-2, i-k-3 \in T$. This situation is enough to construct an example with $s_{i-1} \not\equiv 0$ as we saw while proving the minimality of the second obstruction conditions in the proof of Proposition 2.4.8. In this case, $(\tilde{z}_{i-k-2}, \tilde{z}_{i-k-1}, ..., \tilde{z}_{i-1}) = (p-1, 1, ..., 1)$ and none of the pairs in $\{(i-k-3, i-k-2), (i-k-2, i-k-1), ..., (i-2, i-1)\}$ are transitions.
- $z_{i-k-2} = p-1$ with $i-k-2 \in T$, $i-k-3 \notin T$ and $r_{i-k-3} \not\equiv 0$. This implies that $(\tilde{z}_{i-k-2}, \tilde{z}_{i-k-1}, ..., \tilde{z}_{i-1}) = (1, 1, ..., 1)$ and the sequence is preceded by another sequence that allows $r_{i-k-3} \not\equiv 0$. Moreover the pair $(i-k-3, i-k-2)$ is a transition but none of the pairs in $\{(i-k-2, i-k-1), ..., (i-2, i-1)\}$ are transitions.

By symmetry, similar conditions on $\tilde{z}_j$ exist when $r_{i-1} \not\equiv 0$ mod v.

Combining the analyses for s_{i-1} and r_{i-1} together, we find that whenever there exists an i such that $F_i' \neq U \star F_i$ for all possible choices of U (where U is upper unipotent if F_i is in η-form and lower unipotent if F_i is an η'-form), then $(\tilde{z}_j)_j$ must contain a contiguous subsequence of the form $(p-1, 1, ..., 1, 0)$ of length ≥ 2. On the other hand, if such a contiguous subsequence exists, we can construct an example so that $F_i' \neq U \star F_i$ for some i, for any choice of U (upper or lower unipotent depending on the form of F_i).

Thus, we generalize the definition of second obstruction as follows:

Definition A.3.2. *We say that a tame principal series* $\mathbb{F}$*-type* τ *faces the second obstruction if* $(\tilde{z}_i)_{i\in\mathbb{Z}}$ *contains a contiguous subsequence* $(p-1, 1, ..., 1, 0)$ *of length* ≥ 2.

A.4. **Trivial Serre weight.** The generalizations of the definitions of first and second obstructions (see Definitions A.2.3 and A.3.2) are very similar to the original definitions of first and second obstructions (see Definitions 2.4.7 and 2.4.10). Note that in the case where each Frobenius matrix is in η-form, $\tilde{z}_i = z_i$. By Remark A.1.3, upon requiring τ to not face the first and second obstructions, we exclude no fewer irreducible components of $\mathcal{Z}$ than we had done earlier.

However, notice that the components of $\mathcal{Z}$ indexed by twists of the trivial Serre weight were also not covered under our strategy when we allowed only η-form Frobenius matrices, even though their exclusion did not arise from the first and second obstruction conditions. If $\mathcal{Z}(\sigma)$ is such a component, then by Proposition 4.1.2, the only possible tame principal series $\mathbb{F}$-type $\tau = \eta \oplus \eta'$ such that $\mathcal{C}^{\tau,\mathrm{BT}}(\mathbb{Z}/f\mathbb{Z})$ covers $\mathcal{Z}(\sigma)$ does not satisfy $\eta \neq \eta'$. This situation *can* be rectified by allowing some Frobenius matrices to be in η'-form when $f \geq 2$. By the calculations in Remark A.1.3, all we need is that each $\tilde{z}_i = p-1$, while not all z_i equal 0 (so that $\eta \neq \eta'$). For instance, we can choose $T = \mathbb{Z}/f\mathbb{Z} \smallsetminus \{0\}$, and choose τ so that $z_0 = p-2$, $z_1 = 1$ and all other z_j's equal to $p-1$. A version of Proposition 2.4.12 can be shown to hold for this situation when $p > 3$ and we can find a similar result as in Theorem 5.0.1 when $p > 3$, the Serre weight is trivial and $f \geq 2$. We omit the technical calculations from this paper because the trivial weight is in the Fontaine-Lafaille range and amenable to other methods.

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